Risk Measures on Orlicz Hearts

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June 2007

Abstract

Coherent, convex and monetary risk measures were introduced in a setup where uncertain outcomes are modelled by bounded random variables. In this paper, we study such risk measures on Orlicz hearts. This includes coherent, convex and monetary risk measures on L^p-spaces for $1 \leq p < \infty$ and covers a wide range of interesting examples. Moreover, it allows for an elegant duality theory. We prove that every coherent or convex monetary risk measure on an Orlicz heart which is real-valued on a set with non-empty algebraic interior is real-valued on the whole space and admits a robust representation as maximal penalized expectation with respect to different probability measures. We also show that penalty functions of such risk measures have to satisfy a certain growth condition and that our risk measures are Luxemburg-norm Lipschitz-continuous in the coherent case and locally Luxemburg-norm Lipschitz-continuous in the convex monetary case. In the second part of the paper we investigate cash-additive hulls of transformed Luxemburg-norms and expected transformed losses. They provide two general classes of coherent and convex monetary risk measures that include many of the currently known examples as special cases. Explicit formulas for their robust representations and the maximizing probability measures are given.

Key Words: coherent risk measures, convex monetary risk measures, monetary risk measures, acceptance sets, robust representations, cash-additive hulls, transformed norm risk measures, transformed loss risk measures, Orlicz spaces

1 Introduction

Coherent risk measures were introduced by Artzner et al. (1997, 1999) and extended to more general setups in Delbaen (2000, 2002). Föllmer and Schied (2002a, 2002b, 2004) and Frittelli and Rosazza Gianin (2002) established the more general concepts of convex and monetary risk measures. In Artzner et al. (1997, 1999) and the first part of Föllmer and Schied (2002a), future financial positions are modelled by elements of the set $L(\Omega)$ of all real-valued functions on a finite sample space Ω, and a coherent, convex or monetary risk measure is a mapping $\rho : L(\Omega) \to \mathbb{R}$ satisfying certain properties. In this setting, and expressed in discounted units, the main structural results are that every monetary risk measure ρ can be written as

$$\rho(X) = \inf \{m \in \mathbb{R} : X + m \in \mathcal{C}\} \ , \tag{1.1}$$

We thank Freddy Delbaen, Andreas Hamel, Michael Kupper and Alexander Schied for fruitful discussions and helpful comments.

*Supported by NSF Grant DMS-0642361, a Rheinstein Award and a Peek Fellowship.

for the set of acceptable positions $\mathcal{C} := \{X \in L(\Omega) : \rho(X) \leq 0\}$, and every convex monetary risk measure has a convex dual representation of the form

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{D}} \{\mathrm{E}_{\mathbb{Q}}[-X] - \gamma(\mathbb{Q})\} , \tag{1.2}$$

where $\mathcal{D}$ denotes the set of all probability measures on Ω and γ is a function from $\mathcal{D}$ to $(-\infty, \infty]$. If ρ is coherent, then γ can be chosen so that it only takes the values 0 or ∞, and (1.2) reduces to

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathrm{E}_{\mathbb{Q}}[-X] , \tag{1.3}$$

for the set $\mathcal{Q} := \{\mathbb{Q} \in \mathcal{D} : \gamma(\mathbb{Q}) = 0\}$.

The economic interpretation of (1.1) is that $\rho(X)$ is the minimal amount of cash which has to be added to a position X to make it acceptable. Its proof is elementary, and it easily generalizes to more general setups. (1.2) can be viewed as a robust expectation with respect to different probability measures. The penalty function γ gives different probability measures varying impact in the formula (1.2) because, for example, some of them might be more plausible than others. The standard proofs of the robust representations (1.2) and (1.3) in the literature are based on the separating hyperplane theorem and become more involved in more general frameworks. Also, the form of the representations can slightly change when the set of financial positions $\mathcal{X}$ is different from $L(\Omega)$. So far, the most extensively studied cases have been L^∞ over a probability space and $\mathcal{L}^\infty$ over a measurable space; see, for instance, Delbaen (2000, 2002), Kusuoka (2001), Föllmer and Schied (2002a, 2002b, 2004), Frittelli and Rosazza Gianin (2005), Krätschmer (2005), Jouini et al. (2006). In this case, general robust representations like (1.2) and (1.3) involve finitely additive measures. To reduce them to suprema over σ-additive measures, additional continuity assumptions are needed.

However, most models in finance and insurance mathematics involve unbounded random variables. Therefore, it is natural to study risk measures on bigger sets than L^∞ or $\mathcal{L}^\infty$. In Delbaen (2002) and Cheridito et al. (2006) coherent and convex monetary risk measures on L^0 are investigated and it is shown that to have non-trivial examples, one has to allow them to take values at least in $(-\infty, \infty]$. Frittelli and Rosazza Gianin (2002, 2004) provide robust representations for real-valued risk measures on L^p-spaces. Cherny (2006) studies risk measures on L^0 that take values in $[-\infty, \infty]$ and then introduces subspaces on which they are real-valued. Rockafellar et al. (2006) contains a section on $(-\infty, \infty]$-valued coherent risk measures on L^2 and their relations to deviation measures. Ruszczyński and Shapiro (2006) discuss $[-\infty, \infty]$-valued risk measures on general vector spaces. Delbaen (2006) shows that when a real-valued risk measure is defined on a solid, rearrangement invariant vector space of random variables that contains L^∞, then this space can only contain integrable random variables. For risk measures for unbounded stochastic processes, we refer to Cheridito et al. (2006) and the references therein.

In this paper we study $(-\infty, \infty]$-valued coherent, convex and monetary risk measures on maximal subspaces of Orlicz classes. Following Edgar and Sucheston (1992), we call such spaces Orlicz hearts. They include all L^p-spaces for $1 \leq p < \infty$ and allow for an elegant duality theory without additional continuity assumptions. The structure of the paper is as follows: In Section 2, we introduce the notation and give the basic definitions. In Section 3, we discuss the relation of monetary risk measures for unbounded random variables with their acceptance sets. In Section 4, we show that every coherent or convex monetary risk measure on an Orlicz heart that is real-valued on a set with non-empty algebraic interior is automatically real-valued on the whole space and admits a robust representation of the form (1.3) or (1.2) such that the supremum is attained. We also give the general form of a penalty function in this setup and show that all our risk measures are Luxemburg-norm Lipschitz-continuous in the coherent case and locally Luxemburg-norm Lipschitz-continuous in the convex monetary case. In Section 5, we discuss two general classes of examples. The first one consists of convex monetary risk measures derived from transformed Luxemburg norms, the second one of convex monetary risk measures related to expected transformed losses. Both contain many well-known examples as special cases. We give explicit formulas for their robust representations and the maximizing probability measures.

2 Notation and definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In the whole paper, equalities and inequalities between random variables are understood in the $\mathbb{P}$-almost sure sense. L^0 denotes the space of all real-valued random variables on $(\Omega, \mathcal{F})$, where two random variables are identified if they are $\mathbb{P}$-almost surely equal.

Definition 2.1 *Let $\mathcal{X}$ be a linear subspace of L^0 that contains all constants. We call a mapping $\rho : \mathcal{X} \to (-\infty, \infty]$ a monetary risk measure on $\mathcal{X}$ if it has the following three properties:*

(F) **Finiteness at** 0**:** $\rho(0) \in \mathbb{R}$

(M) **Monotonicity:** $\rho(X) \geq \rho(Y)$ *for all* $X, Y \in \mathcal{X}$ *such that* $X \leq Y$

(T) **Translation property:** $\rho(X + m) = \rho(X) - m$ *for all* $X \in \mathcal{X}$ *and* $m \in \mathbb{R}$

We call a monetary risk measure convex if it also satisfies

(C) **Convexity:** $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ *for all* $X, Y \in \mathcal{X}$ *and* $\lambda \in (0, 1)$

A convex monetary risk measure is called coherent if it fulfills

(P) **Positive homogeneity:** $\rho(\lambda X) = \lambda\rho(X)$ *for all* $X \in \mathcal{X}$ *and* $\lambda \in \mathbb{R}_+$

Elements of $\mathcal{X}$ describe discounted future net worths. The number $\rho(X)$ is understood as a capital requirement for X. Note that (F), (M) and (T) imply that ρ is real-valued on L^∞. (M) means that the capital requirement for X should be greater than for Y if it is clear that X will be smaller than Y in ($\mathbb{P}$-almost) every state of the world. (T) says that adding a constant amount of money m to a position X should reduce the capital requirement for X by m. Under (C), the capital requirement for the convex combination of two positions does not exceed the convex combination of the separate capital requirements. If (P) holds, then capital requirements scale linearly when net worths are multiplied with non-negative constants and (C) is equivalent to

(S) **Subadditivity:** $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{X}$.

The interpretation of (S) is that the capital requirement for an aggregated discounted net worth $X + Y$ should not exceed the sum of the capital requirement for X plus the capital requirement for Y. For a more detailed discussion of the economic interpretations of these axioms we refer to Artzner et al. (1997, 1999), Föllmer and Schied (2002a, 2002b, 2004), and Frittelli and Rosazza Gianin (2002). Relations to pricing and hedging in incomplete markets can be found in Carr et al. (2001), Jaschke and Küchler (2001), and Staum (2004).

3 Acceptance sets

In this section, we give relations of monetary risk measures to their acceptance sets. The situation is mostly parallel to the one in the case of bounded random variables. But some slight adjustments have to be made.

Definition 3.1 *Let $\mathcal{X}$ be a linear subspace of L^0 containing the constants. The acceptance set of a monetary risk measure $\rho : \mathcal{X} \to (-\infty, \infty]$ is given by*

$$\mathcal{C} := \{X \in \mathcal{X} : \rho(X) \leq 0\} .$$

The following two propositions give connections between risk measures and certain sets of random variables. As usual, we use the convention $\inf \emptyset = \infty$.

Proposition 3.2 *Let $\mathcal{X}$ be a linear subspace of L^0 containing the constants and $\rho : \mathcal{X} \to (-\infty, \infty]$ a monetary risk measure with acceptance set $\mathcal{C}$. Then*

$$\rho(X) = \inf \{m \in \mathbb{R} : X + m \in \mathcal{C}\} , \quad X \in \mathcal{X} , \tag{3.1}$$

and $\mathcal{C}$ has the following properties:

(i) $\inf \{m \in \mathbb{R} : m \geq Z \text{ for some } Z \in \mathcal{C}\} \in \mathbb{R}$
(ii) *For all $X \in \mathcal{X}$,* $\inf \{m \in \mathbb{R} : X + m \geq Z \text{ for some } Z \in \mathcal{C}\} \in (-\infty, \infty]$
(iii) *For all $X \in \mathcal{C}$, the set $\{Y \in \mathcal{X} : Y \geq X\}$ is contained in $\mathcal{C}$*
(iv) *If $(X^n)_{n \geq 1}$ is a sequence in $\mathcal{C}$ such that $\|X^n - X\|_\infty \to 0$ for an $X \in \mathcal{X}$, then $X \in \mathcal{C}$*

Moreover, if ρ is convex, then so is $\mathcal{C}$. If ρ is coherent, then $\mathcal{C}$ is a convex cone. If ρ is real-valued, then $\mathcal{C}$ has the following property:

(v) *For all $X \in \mathcal{X}$,* $\inf \{m \in \mathbb{R} : X + m \geq Z \text{ for some } Z \in \mathcal{C}\} \in \mathbb{R}$

Proof. (3.1) follows from the translation property (T). (iii) is a consequence of (M). It implies that for all $X \in \mathcal{X}$,

$$\{m \in \mathbb{R} : X + m \geq Z \text{ for some } Z \in \mathcal{C}\} = \{m \in \mathbb{R} : X + m \in \mathcal{C}\} . \tag{3.2}$$

This together with (3.1) and (F), yields (i) and (ii). As for (iv), if $\|X^n - X\|_\infty \to 0$ for a sequence $(X^n)_{n \geq 1}$ in $\mathcal{C}$, then there exists for all $\varepsilon > 0$ an $n \geq 1$ such that $X \geq X^n - \varepsilon$, and therefore, $\rho(X) \leq \rho(X^n) + \varepsilon \leq \varepsilon$. This shows that X must be in $\mathcal{C}$. That $\mathcal{C}$ inherits convexity from ρ and is a convex cone if ρ is coherent, is obvious. If ρ is real-valued, then (v) follows from (3.1) and (3.2). □

Proposition 3.3 *Let $\mathcal{X}$ be a linear subspace of L^0 containing the constants and $\mathcal{B}$ a subset of $\mathcal{X}$ with the properties* (i) *and* (ii) *of Proposition 3.2. Then*

$$\rho(X) = \inf \{m \in \mathbb{R} : X + m \geq Z \textit{ for some } Z \in \mathcal{B}\}$$

defines a monetary risk measure on $\mathcal{X}$ whose acceptance set $\mathcal{C}$ is the smallest subset of $\mathcal{X}$ that contains $\mathcal{B}$ and satisfies (iii)–(iv) *of Proposition 3.2. If $\mathcal{B}$ is convex, then so is ρ. If $\mathcal{B}$ is a convex cone, then ρ is coherent. If $\mathcal{B}$ satisfies condition* (v) *of Proposition 3.2, then ρ is real-valued.*

Proof. It is clear that ρ is a monetary risk measure and that $\mathcal{B}$ is contained in $\mathcal{C}$. Moreover, for each $X \in \mathcal{C}$ and $n \geq 1$, there exists $Z^n \in \mathcal{B}$ such that $X + 1/n \geq Z^n$. This shows that $\mathcal{C}$ is contained in every subset of $\mathcal{X}$ containing $\mathcal{B}$ and satisfying (iii)–(iv) of Proposition 3.2. That ρ is convex when $\mathcal{B}$ is so, coherent when $\mathcal{B}$ is a convex cone, and real-valued when $\mathcal{B}$ satisfies (v) of Proposition 3.2 is obvious. □

4 Robust representations

In this section we give robust representation results for coherent and convex monetary risk measures on Orlicz hearts that are real-valued on a set with non-empty algebraic interior. An important ingredient in our argumentation is the fact that on a wide class of ordered topological vector spaces, continuity and subdifferentiability of convex functionals can be derived from monotonicity. To different extents this is also exploited for the representation of convex functionals in Cheridito et al. (2004), Ruszczyński and Shapiro (2006), Biagini and Frittelli (2006), Delbaen (2006). Here, we combine it with the special structure of Orlicz hearts to derive a complete dual characterization of coherent and convex monetary risk measures on Orlicz hearts that are real-valued on a set with non-empty algebraic interior.

4.1 Monotone functionals on Banach lattices

We first prove two results for monotone functionals on Banach lattices. The first one only needs monotonicity. The second one shows continuity, subdifferentiability and dual representability for monotone convex functionals. For a function f from a Banach lattice $\mathcal{X}$ to $(-\infty, \infty]$, we denote

$$\operatorname{dom} f := \{x \in \mathcal{X} : f(x) \in \mathbb{R}\} ,$$

and we call f increasing if $f(x) \geq f(y)$ for $x \geq y$. A subset U of $\mathcal{X}$ is an algebraic neighborhood of $x \in \mathcal{X}$ if for every $y \in \mathcal{X}$, there exists an $\varepsilon > 0$ such that $x + ty \in U$ for all $0 \leq t \leq \varepsilon$. The algebraic interior $\operatorname{core}(A)$ of a subset A of $\mathcal{X}$ consists of all $x \in A$ that have an algebraic neighborhood contained in A. In every topological vector space, a neighborhood of x is also an algebraic neighborhood of x. Therefore the interior $\operatorname{int}(A)$ of a subset A of a topological vector space is contained in its algebraic interior $\operatorname{core}(A)$. For increasing functionals on Banach lattices the following holds:

Lemma 4.1 *If f is an increasing function from a Banach lattice $\mathcal{X}$ to $(-\infty, \infty]$, then* $\operatorname{core}(\operatorname{dom} f) = \operatorname{int}(\operatorname{dom} f)$.

Proof. Since $\operatorname{int}(A) \subset \operatorname{core}(A)$ for every subset A of $\mathcal{X}$, we just have to show $\operatorname{core}(\operatorname{dom} f) \subset \operatorname{int}(\operatorname{dom} f)$. By way of contraction, assume that f is real-valued on an algebraic neighborhood of $x \in \mathcal{X}$ but not on a neighborhood of x. Then, there exist elements $y_n \in \mathcal{X}$, $n \geq 1$ with norm $\|y_n\| \leq 4^{-n}$ and $f(x + y_n) = \infty$. The elements y_n^+ still satisfy $\|y_n^+\| \leq 4^{-n}$ and $f(x + y_n^+) = \infty$.

Define $y := \sum_{n\geq 1} 2^n y_n^+$. By assumption, there exists an $\varepsilon > 0$ such that $f(x+\varepsilon y) \in \mathbb{R}$. It follows that for all n with $\varepsilon 2^n \geq 1$, we have

$$\infty > f(x+\varepsilon y) \geq f(x+\varepsilon 2^n y_n^+) \geq f(x+y_n^+) = \infty\,.$$

But this is absurd. So, there has to exist a neighborhood of x on which f is real-valued. □

A convex function f from a topological vector space $\mathcal{X}$ to $[-\infty,\infty]$ is said to be proper if $f(x) < \infty$ for at least one $x \in \mathcal{X}$ and $f(x) > -\infty$ for all $x \in \mathcal{X}$. We call it subdifferentiable at $x \in \mathcal{X}$ if $f(x) \in \mathbb{R}$ and there exists an element x^* in the topological dual $\mathcal{X}^*$ of $\mathcal{X}$ such that $x^*(y) \leq f(x+y) - f(x)$ for all $y \in \mathcal{X}$. For every proper convex function f, the conjugate

$$f^*(x^*) := \sup_{x\in\mathcal{X}} \{x^*(x) - f(x)\}$$

is a $\sigma(\mathcal{X}^*,\mathcal{X})$-lower semicontinuous, convex function from $\mathcal{X}^*$ to $(-\infty,\infty]$. It is immediate from the definition of f^* that

$$f(x) \geq f^{**}(x) := \sup_{x^*\in\mathcal{X}^*} \{x^*(x) - f^*(x^*)\} \quad \text{for all } x \in \mathcal{X}\,.$$

Moreover,

$$f(x) = \max_{x^*\in\mathcal{X}^*} \{x^*(x) - f^*(x^*)\}\,. \tag{4.1}$$

for all $x \in \mathcal{X}$ where f is subdifferentiable.

As a consequence of Lemma 4.1, we get the following refinement of Proposition 3.1 of Ruszczyński and Shapiro (2006):

Theorem 4.2 *Let f be an increasing, convex function from a Banach lattice $\mathcal{X}$ to $(-\infty,\infty]$. Then for all $x \in \operatorname{core}(\operatorname{dom} f)$ the following hold:*

(i) *There exists a neighborhood of x on which f is Lipschitz-continuous with respect to the norm on $\mathcal{X}$*

(ii) *f is subdifferentiable at x*

(iii) $f(x) = \max_{x^*\in\mathcal{X}^*} \{x^*(x) - f^*(x^*)\}$

Proof. By Lemma 4.1, every $x \in \operatorname{core}(\operatorname{dom} f)$ has a neighborhood contained in $\operatorname{dom} f$. So it follows from Proposition 3.1 of Ruszczyński and Shapiro (2006) that f is continuous and subdifferentiable at x. This proves (ii). (iii) is a consequence of (ii) and (4.1). Since f is continuous at x, there exists a neighborhood of x on which f is bounded and (i) follows from Corollary 2.2.12 in Zălinescu (2002). □

4.2 Robust representation of risk measures on Orlicz hearts

We now consider risk measures on Orlicz hearts. We shortly review some basic facts of Orlicz space theory. We call a function $\Phi : [0,\infty) \to [0,\infty]$ a Young function if it is left-continuous, convex, $\lim_{x\downarrow 0} \Phi(x) = \Phi(0) = 0$, and $\lim_{x\to\infty} \Phi(x) = \infty$. It follows from these properties that Φ is increasing, by which we mean that $\Phi(x) \geq \Phi(y)$ for $x \geq y$. Also, Φ is continuous except possibly at a single point, where it jumps to ∞. So, the assumption of left-continuity is needed only at that one point. The conjugate

$$\Psi(y) := \sup_{x\geq 0} \{xy - \Phi(x)\}\,, \quad y \geq 0\,.$$

is again a Young function, and its conjugate is Φ. The Orlicz heart corresponding to Φ is given by

$$M^\Phi := \left\{ X \in L^\Phi : \mathrm{E}_\mathbb{P}\left[\Phi(c\,|X|)\right] < \infty \text{ for all } c > 0 \right\} .$$

It is the largest linear subspace contained in the Orlicz class

$$\tilde{L}^\Phi := \left\{ X \in L^0 : \mathrm{E}_\mathbb{P}\left[\Phi(|X|)\right] < \infty \right\} ,$$

which is a convex subset of the Orlicz space

$$L^\Phi := \left\{ X \in L^0 : \mathrm{E}_\mathbb{P}\left[\Phi(c\,|X|)\right] < \infty \text{ for some } c > 0 \right\} .$$

The Luxemburg norm

$$\|X\|_\Phi := \inf\left\{ \lambda > 0 : \mathrm{E}\left[\Phi\left(\left|\frac{X}{\lambda}\right|\right)\right] \leq 1 \right\}$$

and the Orlicz norm

$$\|X\|^*_\Psi := \sup\left\{ \mathrm{E}_\mathbb{P}\left[XY\right] : \|Y\|_\Psi \leq 1 \right\}$$

are equivalent norms on L^Φ under which L^Φ with the $\mathbb{P}$-almost sure ordering $\geq$ is a Banach lattice.

Note that if Φ jumps to ∞, then M^Φ is equal to the trivial space $\{0\}$. So, from now on, we assume that Φ is real-valued. Then, Φ is continuous, M^Φ is the $\|.\|_\Phi$-closure of L^∞ in L^Φ, and the norm dual of $(M^\Phi, \|.\|_\Phi)$ is given by $(L^\Psi, \|.\|^*_\Phi)$, where $Y \in L^\Psi$ acts on $X \in M^\Phi$ by $Y(X) := \mathrm{E}_\mathbb{P}\left[XY\right]$. For a proof of the last two statements, we refer to Theorems 2.1.14 and 2.2.11 in Edgar and Sucheston (1992).

If the right-sided derivative Φ'_+ is bounded from above by a constant $C > 0$, then $\Psi(y) = \infty$ for all $y > C$, $M^\Phi = L^\Phi = L^1$ and $L^\Psi = L^\infty$. If Φ'_+ is unbounded, then $\Psi(y) < \infty$ for all $y \geq 0$. In particular, Ψ is continuous and $L^\infty \subset M^\Phi, L^\Psi \subset L^1$.

For illustration, we give three simple examples:

Examples 4.3

1. For $\Phi(x) = x$, we have

$$\Psi(y) = \begin{cases} 0 & \text{for } y \leq 1 \\ \infty & \text{for } y > 1 \end{cases} ,$$

and

$$M^\Phi = L^\Phi = L^1 , \quad \|.\|_\Phi = \|.\|_1 , \quad L^\Psi = L^\infty , \quad \|.\|^*_\Phi = \|.\|_\infty .$$

2. If $\Phi(x) = x^p$ for $p \in (1, \infty)$, then $\Psi(y) = p^{1-q} q^{-1} y^q$, and we have

$$M^\Phi = L^\Phi = L^p , \quad \|.\|_\Phi = \|.\|_p , \quad L^\Psi = L^q , \quad \|.\|^*_\Phi = \|.\|_q .$$

3. If $\Phi(x) = \exp(\lambda x) - 1$ for $\lambda > 0$, then

$$\Psi(y) = \begin{cases} 0 & \text{for } y \leq \lambda \\ \frac{y}{\lambda}\log\left(\frac{y}{\lambda}\right) - \frac{y}{\lambda} + 1 & \text{for } y > \lambda \end{cases} ,$$

and $L^\infty \subset M^\Phi \subset L^p \subset L^\Psi \subset L^1$ for all $p \in (1, \infty)$.

In the following we identify a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ that is absolutely continuous with respect to $\mathbb{P}$ with its Radon–Nikodym derivative $\xi = d\mathbb{Q}/d\mathbb{P} \in L^1$. Then, the set

$$\mathcal{D} := \left\{ \xi \in L^1 \mid \xi \geq 0 \, , \, \mathrm{E}_{\mathbb{P}}[\xi] = 1 \right\}$$

represents all probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to $\mathbb{P}$. By $\mathcal{D}^\Psi$ we denote the intersection $\mathcal{D} \cap L^\Psi$.

Definition 4.4 *We call a mapping $\gamma : \mathcal{D}^\Psi \to (-\infty, \infty]$ a penalty function on $\mathcal{D}^\Psi$ if it is bounded from below and not identically equal to ∞. We say that γ satisfies the growth condition* (G) *if there exist constants $a \in \mathbb{R}$ and $b > 0$ such that*

$$\gamma(\mathbb{Q}) \geq a + b \, \|\mathbb{Q}\|^*_\Phi \quad \textit{for all } \mathbb{Q} \in \mathcal{D}^\Psi \, .$$

For any penalty function γ on $\mathcal{D}^\Psi$, we define

$$\rho_\gamma(X) := \sup_{\mathbb{Q} \in \mathcal{D}^\Psi} \left\{ \mathrm{E}_{\mathbb{Q}}[-X] - \gamma(\mathbb{Q}) \right\} \, , \quad X \in M^\Phi$$

and call it a robust representation of ρ_γ.

It can easily be checked that ρ_γ defines a lower semicontinuous convex monetary risk measure on M^Φ with values in $(-\infty, \infty]$. For penalty functions satisfying the growth condition (G) we have the following result:

Theorem 4.5 *Let γ be a penalty function on $\mathcal{D}^\Psi$. Then the following three conditions are equivalent:*

(i) *γ satisfies the growth condition* (G)

(ii) $\mathrm{core}(\mathrm{dom}\, \rho_\gamma) \neq \emptyset$

(iii) *ρ_γ is real-valued and every $X \in M^\Phi$ has a neighborhood on which ρ_γ is Lipschitz-continuous with respect to $\|.\|_\Phi$*

Proof. We show (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). The first implication is trivial.

To prove (ii) $\Rightarrow$ (i), assume that ρ_γ is real-valued on an algebraic neighborhood of $X \in M^\Phi$. Since the mapping $Y \mapsto \rho_\gamma(-Y)$ is increasing, we obtain from Lemma 4.1 that there exists an $\varepsilon > 0$ such that ρ_γ is real-valued on the closed ball $B_\varepsilon(X)$ with radius ε around X. Since L^∞ is $\|.\|_\Phi$-dense in M^Φ, there exists a sequence $(Y^n)_{n \geq 1}$ of bounded random variables such that $\|Y^n - X\|_\Phi \leq \varepsilon \, 2^{-n-2}$. If γ does not satisfy the growth condition (G), then there exists a sequence of probability measures $(\mathbb{Q}^n)_{n \geq 1}$ in $\mathcal{D}^\Psi$ such that

$$\gamma(\mathbb{Q}^n) < -n - \|Y^n\|_\infty + \varepsilon \, 2^{-n-2} \, \|\mathbb{Q}^n\|^*_\Phi \quad \text{for all } n \geq 1 \, .$$

Since $(L^\Psi, \|.\|^*_\Phi)$ is the norm dual of $(M^\Phi, \|.\|_\Phi)$, there exists for every $n \geq 1$, $Z^n \in M^\Phi$ such that $Z^n \leq 0$, $\|Z^n\|_\Phi \leq 1$ and $\mathrm{E}_{\mathbb{Q}^n}[-Z^n] \geq \frac{1}{2} \|\mathbb{Q}^n\|^*_\Phi$. The random variable $Z := \varepsilon \sum_{n \geq 1} 2^{-n} Z^n$ is in M^Φ with norm $\|Z\|_\Phi \leq \varepsilon$, and

$$\begin{aligned}
& \rho_\gamma(X + Z) \geq \rho_\gamma(X + \varepsilon 2^{-n} Z^n) \geq \mathrm{E}_{\mathbb{Q}^n}\left[-X - \varepsilon 2^{-n} Z^n\right] - \gamma(\mathbb{Q}^n) \\
\geq \; & \mathrm{E}_{\mathbb{Q}^n}[-Y^n] + \mathrm{E}_{\mathbb{Q}^n}[Y^n - X] + \varepsilon \, 2^{-n} \, \mathrm{E}_{\mathbb{Q}^n}[-Z^n] + n + \|Y^n\|_\infty - \varepsilon \, 2^{-n-2} \, \|\mathbb{Q}^n\|^*_\Phi \\
\geq \; & -\|Y^n\|_\infty - \|Y^n - X\|_\Phi \, \|\mathbb{Q}^n\|^*_\Phi + \varepsilon \, 2^{-n-1} \, \|\mathbb{Q}^n\|^*_\Phi + n + \|Y^n\|_\infty - \varepsilon \, 2^{-n-2} \, \|\mathbb{Q}^n\|^*_\Phi \\
\geq \; & n \quad \text{for all } n \geq 1 \, .
\end{aligned}$$

But this contradicts the finiteness of ρ_γ on $B_\varepsilon(X)$. Therefore, γ must fulfill the growth condition (G), and (i) is proved.

(i) $\Rightarrow$ (iii): Assume there exist constants $a \in \mathbb{R}$ and $b > 0$ such that $\gamma(\mathbb{Q}) \geq a + b \, \|\mathbb{Q}\|_\Phi^*$ for all $\mathbb{Q} \in \mathcal{D}^\Psi$. Choose $X \in M^\Phi$. There exists $\hat{X} \in L^\infty$ with $\|X - \hat{X}\|_\Phi \leq b$, and we obtain

$$\begin{aligned} & \mathrm{E}_{\mathbb{Q}}[-X] - \gamma(\mathbb{Q}) = \mathrm{E}_{\mathbb{Q}}[-\hat{X}] + \mathrm{E}_{\mathbb{Q}}[\hat{X} - X] - \gamma(\mathbb{Q}) \\ \leq \quad & \|\hat{X}\|_\infty + \|\hat{X} - X\|_\Phi \, \|\mathbb{Q}\|_\Phi^* - a - b \, \|\mathbb{Q}\|_\Phi^* \leq \|\hat{X}\|_\infty - a \end{aligned}$$

for all $\mathbb{Q} \in \mathcal{D}^\Psi$. This shows that $\rho_\gamma(X) \leq \|\hat{X}\|_\infty - a < \infty$. Hence, ρ_γ is real-valued. The rest of (iii) follows from Theorem 4.2(i). □

In the next theorem we show that every convex monetary risk measure ρ on M^Φ with $\mathrm{core}(\mathrm{dom}\,\rho) \neq \emptyset$ is of the form ρ_γ for a penalty function γ, and that the minimal penalty function of ρ is given by

$$\rho^{\#}(\mathbb{Q}) := \sup_{X \in M^\Phi} \left\{ \mathrm{E}_{\mathbb{Q}}[-X] - \rho(X) \right\} , \quad \mathbb{Q} \in \mathcal{D}^\Psi . \tag{4.2}$$

Note that $\rho^{\#}(\mathbb{Q}) = f^*(\mathbb{Q})$ for all $\mathbb{Q} \in \mathcal{D}^\Psi$, where f is the increasing, convex function given by $f(X) = \rho(-X)$. Since ρ satisfies (M) and (T), one obtains $f^*(\xi) = \infty$ for all $\xi \in L^\Psi \setminus \mathcal{D}^\Psi$.

Theorem 4.6 *Let $\rho : M^\Phi \to (-\infty, \infty]$ be a convex monetary risk measure with* $\mathrm{core}(\mathrm{dom}\,\rho) \neq \emptyset$*. Then ρ is real-valued, $\rho^{\#}$ is a penalty function on $\mathcal{D}^\Psi$ satisfying the growth condition* (G)*, and*

$$\rho(X) = \max_{\mathbb{Q} \in \mathcal{Q}^\Psi} \left\{ \mathrm{E}_{\mathbb{Q}}[-X] - \rho^{\#}(\mathbb{Q}) \right\} \quad \textit{for all } X \in M^\Phi . \tag{4.3}$$

Moreover, if $\rho = \rho_\gamma$ for a penalty function γ on $\mathcal{D}^\Psi$, then $\rho^{\#}$ is the greatest convex, $\sigma(L^\Psi, M^\Phi)$-lower semicontinuous minorant of γ.

Proof. The function $f(X) = \rho(-X)$ is increasing and convex. Since $f^*(\xi) = \infty$ for all $\xi \in L^\Psi \setminus D^\Psi$, it follows from Theorem 4.2(iii) that

$$\rho(X) = f(-X) = \max_{\mathbb{Q} \in \mathcal{D}^\Psi} \left\{ \mathrm{E}_{\mathbb{Q}}[-X] - f^*(\mathbb{Q}) \right\} = \max_{\mathbb{Q} \in \mathcal{D}^\Psi} \left\{ \mathrm{E}_{\mathbb{Q}}[-X] - \rho^{\#}(\mathbb{Q}) \right\} \tag{4.4}$$

for all $X \in \mathrm{core}(\mathrm{dom}\,\rho)$. This shows that ρ has a continuous affine minorant. Therefore, the greatest lower semicontiuous minorant $\bar{\rho}$ of ρ is proper, and we obtain from Theorem 2.3.4 in Zălinescu (2002) that

$$\bar{\rho}(X) = \sup_{\mathbb{Q} \in \mathcal{D}^\Psi} \left\{ \mathrm{E}_{\mathbb{Q}}[-X] - \rho^{\#}(\mathbb{Q}) \right\} \quad \text{for all } X \in M^\Phi . \tag{4.5}$$

Since $-\infty < \bar{\rho}(0) \leq \rho(0) < \infty$, it follows from (4.5) that $\rho^{\#}$ is a penalty function. Furthermore, $\mathrm{core}(\mathrm{dom}\,\bar{\rho}) \supset \mathrm{core}(\mathrm{dom}\,\rho) \neq \emptyset$. So Theorem 4.5 yields that $\bar{\rho}$ is real-valued and $\rho^{\#}$ satisfies the growth condition (G). This implies that the convex set $\mathrm{dom}\,\rho$ is dense in M^Φ. By Lemma 4.1, it has non-empty interior. Now assume that there exists $Y \in M^\Phi \setminus \mathrm{dom}\,\rho$. Then, by Eidelheit's separation theorem (see, e.g., Theorem 1.1.3 in Zălinescu, 2002), there exists $\xi \in L^\Psi \setminus \{0\}$ and $Z \in M^\Phi$ such that

$$\sup_{X \in \mathrm{dom}\,\rho} \mathrm{E}_{\mathbb{P}}[X\xi] \leq \mathrm{E}_{\mathbb{P}}[Y\xi] < \mathrm{E}_{\mathbb{P}}[Z\xi] .$$

But this contradicts the density of $\operatorname{dom}\rho$ in M^Φ. Hence, ρ must be real-valued. The representation (4.3) now follows from (4.4).

To prove the last part of the theorem, let γ be a penalty function on $\mathcal{D}^\Psi$ with $\rho = \rho_\gamma$. Denote by $\hat{\gamma}$ the function from L^Ψ to $(-\infty, \infty]$ which is equal to γ on $\mathcal{D}^\Psi$ and ∞ on $L^\Psi \setminus \mathcal{D}^\Psi$. Then f^* is the biconjugate of $\hat{\gamma}$ in the duality (L^Ψ, M^Φ). Since $\hat{\gamma}$ is bounded from below, it follows from Theorem 2.3.4 in Zălinescu (2002) that f^* is the greatest convex, $\sigma(L^\Psi, M^\Phi)$-lower semicontinuous minorant of $\hat{\gamma}$. Since $\rho^\#$ is the restriction of f^* to $\mathcal{D}^\Psi$, this completes the proof. □

For every non-empty subset $\mathcal{Q}$ of $\mathcal{D}^\Psi$,

$$\gamma(\mathbb{Q}) = \begin{cases} 0 & \text{for } \mathbb{Q} \in \mathcal{Q} \\ \infty & \text{for } \mathbb{Q} \notin \mathcal{Q} \end{cases}$$

is a penalty function on $\mathcal{D}^\Psi$. It is easy to see that the corresponding risk measure

$$\rho_{\mathcal{Q}}(X) := \sup_{\mathbb{Q} \in \mathcal{Q}} \mathrm{E}_{\mathbb{Q}}[-X]$$

is coherent, and γ fulfills the growth condition (G) if and only if $\mathcal{Q}$ is $\|.\|_\Phi^*$-bounded. Hence, we obtain the following corollary to Theorem 4.5:

Corollary 4.7 *Let $\mathcal{Q}$ be a non-empty subset of $\mathcal{D}^\Psi$. Then the following three conditions are equivalent:*
(i) *$\mathcal{Q}$ is $\|.\|_\Phi^*$-bounded*
(ii) $\operatorname{core}(\operatorname{dom}\rho_{\mathcal{Q}}) \neq \emptyset$
(iii) *$\rho_{\mathcal{Q}}$ is real-valued and Lipschitz-continuous with respect to $\|.\|_\Phi$*

Proof. The only statement of the corollary that is not an immediate consequence of Theorem 4.5 is the Lipschitz-continuity of $\rho_{\mathcal{Q}}$ when (i) or (ii) holds. But for all $X, Y \in M^\Phi$ and $\mathbb{Q} \in \mathcal{Q}$,

$$\rho_{\mathcal{Q}}(X) \leq \mathrm{E}_{\mathbb{Q}}[Y - X] + \mathrm{E}_{\mathbb{Q}}[-Y] \leq \|X - Y\|_\Phi \|\mathbb{Q}\|_\Phi^* + \rho_{\mathcal{Q}}(Y)$$

and analogously,

$$\rho_{\mathcal{Q}}(Y) \leq \|Y - X\|_\Phi \|\mathbb{Q}\|_\Phi^* + \rho_{\mathcal{Q}}(X) .$$

So if (i) holds, then $K := \sup_{\mathbb{Q} \in \mathcal{Q}} \|\mathbb{Q}\|_\Phi^*$ is finite, and

$$|\rho_{\mathcal{Q}}(X) - \rho_{\mathcal{Q}}(Y)| \leq K \|X - Y\|_\Phi .$$

□

If ρ is a coherent risk measure on M^Φ, it follows from the positive homogeneity of ρ that $\rho^\#(\mathbb{Q}) = 0$ if

$$\mathrm{E}_{\mathbb{Q}}[X] + \rho(X) \geq 0 \text{ for all } X \in M^\Psi \tag{4.6}$$

and $\rho^\#(\mathbb{Q}) = \infty$ otherwise. Since $\rho(X + \rho(X)) = 0$ for all $X \in M^\Phi$, condition (4.6) is equivalent to $\mathrm{E}_{\mathbb{Q}}[X] \geq 0$ for all X in the acceptance set $\mathcal{C} = \{X \in M^\Phi : \rho(X) \leq 0\}$. Hence, Theorem 4.6 reduces to the following statement for coherent risk measures on M^Φ:

Corollary 4.8 *Let $\rho : M^\Phi \to (-\infty, \infty]$ be a coherent risk measure with acceptance set $\mathcal{C}$. If* core(dom ρ) $\neq \emptyset$, *then ρ is real-valued and can be represented as*

$$\rho(X) = \max_{\mathbb{Q} \in \mathcal{Q}} \mathrm{E}_{\mathbb{Q}}[-X] , \quad X \in M^\Phi , \tag{4.7}$$

for the $\|.\|^*_\Phi$-bounded, convex set

$$\mathcal{Q} := \left\{ \mathbb{Q} \in \mathcal{D}^\Psi : \mathrm{E}_{\mathbb{Q}}[X] \geq 0 \text{ for all } X \in \mathcal{C} \right\} .$$

Moreover, if $\mathcal{R}$ is a subset of $\mathcal{D}^\Psi$ such that $\rho = \rho_{\mathcal{R}}$, then $\mathcal{Q}$ is the $\sigma(L^\Psi, M^\Phi)$-closed, convex hull of $\mathcal{R}$.

5 Examples

In this section, we investigate two classes of convex monetary risk measures on Orlicz hearts. Members of the first class are derived from transformed Luxemburg norms, those of the second class from expected transformed losses.

5.1 Cash-additive hulls

All risk measures in this section are cash-additive hulls of convex functionals, a structure that has also appeared in Ben-Tal and Teboulle (1986, 1987), Krokhmal and Murphey (2006), Filipović and Kupper (2007). We shortly recapitulate what we need to know about it.

Let V be a mapping from M^Φ to $(-\infty, \infty]$ with the following three properties:

(V1) $V(X) \geq V(Y)$ for all $X, Y \in M^\Phi$ such that $X \geq Y$
(V2) $V(\lambda X + (1-\lambda)Y) \leq \lambda V(X) + (1-\lambda)V(Y)$ for all $X, Y \in M^\Phi$ and $\lambda \in (0,1)$
(V3) $\inf_{s \in \mathbb{R}} \{V(s - X) - s\} \in \mathbb{R}$ for all $X \in M^\Phi$.

Then, it can easily be checked that

$$\rho^V(X) := \inf_{s \in \mathbb{R}} \{V(s - X) - s\} \tag{5.1}$$

is the largest real-valued convex monetary risk measures on M^Φ such that

$$\rho^V(X) \leq V(-X) \quad \text{for all } X \in M^\Phi .$$

We call it the cash-additive hull of the decreasing convex functional $V(-.)$. It is immediate from definition (4.2) that the minimal penalty function of ρ^V is given by

$$\begin{aligned} \left(\rho^V\right)^{\#}(\mathbb{Q}) &= \sup_{X \in M^\Phi} \left\{ \mathrm{E}_{\mathbb{Q}}[-X] - \rho^V(X) \right\} = \sup_{X \in M^\Phi, s \in \mathbb{R}} \left\{ \mathrm{E}_{\mathbb{Q}}[-X] - V(s - X) + s \right\} \\ &= \sup_{X \in M^\Phi} \left\{ \mathrm{E}_{\mathbb{Q}}[X] - V(X) \right\} . \end{aligned} \tag{5.2}$$

In fact, all examples we discuss in this section satisfy the following stronger variant of (V3):

(V3') $\min_{s \in \mathbb{R}} \{V(s - X) - s\} \in \mathbb{R}$ for all $X \in M^\Phi$.

Then, (5.1) becomes

$$\rho^V(X) := \min_{s \in \mathbb{R}} \{V(s - X) - s\} . \tag{5.3}$$

5.2 Transformed norm risk measures

Let F be a left-continuous, increasing, convex function from $[0,\infty)$ to $(-\infty,\infty]$ such that $\lim_{x\to\infty} F(x)=\infty$, G a real-valued Young function, and $H:\mathbb{R}\to[0,\infty)$ an increasing, convex function with $\lim_{x\to\infty} H(x)=\infty$. We assume that the following two conditions are satisfied:

$$\text{(FGH1)} \qquad F\left(\frac{H(x)+\varepsilon}{G^{-1}(1)}\right)<\infty \quad \text{for some } x\in\mathbb{R} \text{ and } \varepsilon>0$$
$$\text{(FGH2)} \qquad \lim_{x\to\infty}\left\{F\circ H(x)-G^{-1}(1)\,x\right\}=\infty\,.$$

Note that G and H are real-valued and continuous while F can jump to ∞. Define

$$H_0(x):=H(x)-H(0)\,,\quad x\geq 0\,.$$

Then $\Phi:=G\circ H_0$ is a real-valued Young function. In the next two lemmas we show that

$$X\mapsto F\left(\|H(X)\|_G\right) \tag{5.4}$$

is a well-defined mapping from M^Φ to $(-\infty,\infty]$ with the properties (V1)–(V3'). Therefore,

$$\mathrm{T}(X)=\min_{s\in\mathbb{R}}\left\{F\left(\|H(s-X)\|_G\right)-s\right\} \tag{5.5}$$

defines a real-valued convex monetary risk measure on M^Φ. We call it a transformed norm risk measure.

Lemma 5.1
(i) $H(X)\in M^G$ *for all* $X\in M^\Phi$.
(ii) *If* $X\in M^\Phi$ *and* $\|H(X)\|_G\geq 1+H(0)/G^{-1}(1)$, *then* $\|H(X)\|_G\geq\|X^+\|_\Phi$.

Proof. (i) Let $X\in M^\Phi$. Since H_0 is convex and $H_0(0)=0$, we have

$$H_0(\lambda x)\leq\lambda H_0(x)\quad\text{for all } x\geq 0 \text{ and } 0\leq\lambda\leq 1\,. \tag{5.6}$$

Therefore,

$$\mathrm{E}_\mathbb{P}\left[G(cH_0(|X|))\right]\leq\mathrm{E}_\mathbb{P}\left[G(H_0(c|X|))\right]=\mathrm{E}_\mathbb{P}\left[\Phi(c\,|X|)\right]<\infty\quad\text{for all } c\geq 1\,.$$

This shows that $H_0(|X|)$ belongs to M^G. Since M^G contains the constants and

$$0\leq H(X)\leq H(|X|)=H_0(|X|)+H(0)\,,$$

we also get $H(X)\in M^G$.

(ii) Let $X\in M^\Phi$ with $\|H(X)\|_G\geq 1+H(0)/G^{-1}(1)$. Then

$$\left\|H_0(X^+)\right\|_G\geq\|H(X)\|_G-\|H(0)\|_G=\|H(X)\|_G-\frac{H(0)}{G^{-1}(1)}\geq 1\,.$$

Hence, we obtain from (5.6),

$$\begin{aligned}1 &= \mathrm{E}_\mathbb{P}\left[G\left(\frac{H_0(X^+)}{\|H_0(X^+)\|_G}\right)\right]\geq\mathrm{E}_\mathbb{P}\left[G\circ H_0\left(\frac{X^+}{\|H_0(X^+)\|_G}\right)\right]\\ &= \mathrm{E}_\mathbb{P}\left[\Phi\left(\frac{X^+}{\|H_0(X^+)\|_G}\right)\right],\end{aligned}$$

which shows that

$$\left\|X^+\right\|_\Phi \le \left\|H_0(X^+)\right\|_G \le \|H(X)\|_G \,.$$

□

Lemma 5.1(i) shows that (5.4) is a well-defined mapping from M^Φ to $(-\infty,\infty]$. It clearly fulfills (V1) and (V2). In the next lemma we prove that it also satisfies (V3').

Lemma 5.2 *Assume* (FGH1) *and* (FGH2). *Then there exists for all $X \in M^\Phi$ an $s_X \in \mathbb{R}$ such that*

$$F\left(\|H(s_X - X)\|_G\right) - s_X = \inf_{s\in\mathbb{R}} \left\{F\left(\|H(s-X)\|_G\right) - s\right\} \in \mathbb{R}\,. \tag{5.7}$$

Proof. Let $X \in M^\Phi$. By (FGH1), there exists $x \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$F\left(\|H(x)+\varepsilon\|_G\right) = F\left(\frac{H(x)+\varepsilon}{G^{-1}(1)}\right) < \infty\,.$$

Since $\lim_{s\to-\infty} \|H(s-X)\|_G = \|H(-\infty)\|_G$, there exists $s_0 \in \mathbb{R}$ such that $\|H(s_0 - X)\|_G \le \|H(x)+\varepsilon\|_G$, and therefore, $F\left(\|H(s_0-X)\|_G\right) < \infty$. This shows that

$$k(s) := F\left(\|H(s-X)\|_G\right) - s$$

is a left-continuous, convex function from $\mathbb{R}$ to $(-\infty,\infty]$ such that $k(s) \in \mathbb{R}$ for all $s \le s_0$. Clearly, $F\left(\|H(s-X)\|_G\right) \ge F(0) > -\infty$ for all $s \in \mathbb{R}$. Therefore, $\lim_{s\to-\infty} k(s) = \infty$. If we can also show

$$\lim_{s\to\infty} k(s) = \infty\,, \tag{5.8}$$

then there must exist a minimizer s_X of $k(s)$ and the lemma follows. Since F and H are non-constant, increasing, convex functions, we have

$$\lim_{x\to\infty} \frac{F(x)}{x} = a \quad \text{and} \quad \lim_{x\to\infty} \frac{H(x)}{x} = b \quad \text{for } a,b \in (0,\infty]\,.$$

It follows from assumption (FGH2) that $ab > G^{-1}(1)$. Therefore, there exists $x_0 \in \mathbb{R}$ such that $ab > G^{-1}\left(1/\mathbb{P}[A]\right)$ for $A := \{X \le x_0\}$. Then,

$$F\left(\|H(s-X)\|_G\right) \ge F\left(\|1_A H(s-x_0)\|_G\right) = F\left(\frac{H(s-x_0)}{G^{-1}(1/\mathbb{P}[A])}\right) \quad \text{for all } s\,,$$

and

$$\lim_{s\to\infty} \frac{1}{s} F\left(\frac{H(s-x_0)}{G^{-1}(1/\mathbb{P}[A])}\right) = \frac{ab}{G^{-1}(1/\mathbb{P}[A])} > 1\,.$$

This implies (5.8) and proves the lemma. □

By Lemmas 5.1 and 5.2, the specification (5.4) of V satisfies (V1)–(V3') on M^Φ. Therefore, T is a real-valued convex monetary risk measure on M^Φ. In the following theorem we derive the minimal penalty function $\mathrm{T}^\#$ of T. For this we need the convex conjugates

$$\begin{aligned}
F^*(y) &:= \sup_{x\ge0} \{xy - F(x)\}\,, \quad y \ge 0\\
G^*(y) &:= \sup_{x\ge0} \{xy - G(x)\}\,, \quad y \ge 0\\
H^*(y) &:= \sup_{x\in\mathbb{R}} \{xy - H(x)\}\,, \quad y \ge 0\,.
\end{aligned}$$

Note that G^* is a Young function with corresponding Orlicz space L^{G^*}.

Theorem 5.3 *Under the assumptions* (FGH1) *and* (FGH2), T *is a real-valued convex monetary risk measure on* M^Φ *with minimal penalty function*

$$\mathrm{T}^{\#}(\mathbb{Q}) \quad = \quad \min_{\eta \in L_+^{G^*},\, \eta \gg \mathbb{Q}} \left\{ \mathrm{E}_{\mathbb{P}} \left[\eta\, H^* \left(\frac{1}{\eta} \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] + F^*\left(\|\eta\|_G^* \right) \right\}, \tag{5.9}$$

where $\eta \gg \mathbb{Q}$ *means* $\{\eta = 0\} \subset \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} = 0 \right\}$ *and fractions of the form* $\frac{1}{\eta}\frac{d\mathbb{Q}}{d\mathbb{P}}$ *are understood to be* 0 *on* $\left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} = 0 \right\}$.

Proof. T equals ρ^V for $V(X) = F\left(\|H(X)\|_G\right)$, which by Lemmas 5.1 and 5.2, satisfies (V1)–(V3') on M^Φ. Therefore, T is a real-valued convex monetary risk measure on M^Φ.

To derive the form of the minimal penalty function $\mathrm{T}^{\#}$ of T, we fix $\mathbb{Q} \in \mathcal{D}^\Psi$. By Lemma 5.1(i), $H(X)$ belongs to M^G for all $X \in M^\Phi$. By Theorem 2.2.11 of Edgar and Sucheston (1992), the norm dual of $(M^G, \|.\|_G)$ is $(L^{G^*}, \|.\|_G^*)$. Therefore, we obtain from the Hahn–Banach theorem

$$\|H(X)\|_G = \max_{\eta \in L_+^{G^*},\, \|\eta\|_G^* = 1} \mathrm{E}_{\mathbb{P}}\left[H(X)\,\eta\right] \quad \text{for all } X \in M^\Phi\,,$$

where $L_+^{G^*} := \left\{ \eta \in L^{G^*} : \eta \geq 0 \right\}$. Moreover, the left-continuous, increasing, convex function F can be written as

$$F(x) = \sup_{y \geq 0} \left\{ xy - F^*(y) \right\}, \quad x \geq 0\,.$$

Thus, by the general formula (5.2),

$$\begin{aligned}
\mathrm{T}^{\#}(\mathbb{Q}) &= \sup_{X \in M^\Phi} \left\{ \mathrm{E}_{\mathbb{Q}}[X] - F\left(\|H(X)\|_G\right) \right\} \\
&= \sup_{X \in M^\Phi} \; \inf_{y \geq 0,\, \eta \in L_+^{G^*},\, \|\eta\|_G^* = 1} \left\{ \mathrm{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} X \right] - y \mathrm{E}_{\mathbb{P}}\left[H(X)\,\eta\right] + F^*(y) \right\} \\
&= \sup_{X \in M^\Phi} \; \inf_{\eta \in L_+^{G^*}} \left\{ \mathrm{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} X - H(X)\,\eta \right] + F^*\left(\|\eta\|_G^*\right) \right\} \\
&\leq \inf_{\eta \in L_+^{G^*}} \; \sup_{X \in M^\Phi} \left\{ \mathrm{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} X - H(X)\,\eta \right] + F^*\left(\|\eta\|_G^*\right) \right\}.
\end{aligned} \tag{5.10}$$

If we can show that the infimum in (5.10) is a minimum and

$$\mathrm{T}^{\#}(\mathbb{Q}) = \min_{\eta \in L_+^{G^*}} \; \sup_{X \in M^\Phi} \left\{ \mathrm{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} X - H(X)\,\eta \right] + F^*\left(\|\eta\|_G^*\right) \right\}, \tag{5.11}$$

then we can derive formula (5.9) as follows: First, it is easy to see that for every $\eta \in L_+^{G^*}$ with $\mathbb{P}\left[\eta = 0 \text{ and } \frac{d\mathbb{Q}}{d\mathbb{P}} > 0\right] > 0$, the supremum in (5.11) is ∞. Therefore, the minimum in (5.11) can be taken over all $\eta \in L_+^{G^*}$ such that $\eta \gg \mathbb{Q}$. Thus,

$$\begin{aligned}
\mathrm{T}^{\#}(\mathbb{Q}) &= \min_{\eta \in L_+^{G^*},\, \eta \gg \mathbb{Q}} \; \sup_{X \in M^\Phi} \left\{ \mathrm{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} X - H(X)\,\eta \right] + F^*\left(\|\eta\|_G^*\right) \right\} \\
&= \min_{\eta \in L_+^{G^*},\, \eta \gg \mathbb{Q}} \; \sup_{X \in M^\Phi} \left\{ \mathrm{E}_{\mathbb{P}} \left[\eta \left(\frac{1}{\eta} \frac{d\mathbb{Q}}{d\mathbb{P}} X - H(X) \right) \right] + F^*\left(\|\eta\|_G^*\right) \right\} \qquad (5.12) \\
&= \min_{\eta \in L_+^{G^*},\, \eta \gg \mathbb{Q}} \left\{ \mathrm{E}_{\mathbb{P}} \left[\eta\, H^* \left(\frac{1}{\eta} \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] + F^*\left(\|\eta\|_G^*\right) \right\},
\end{aligned}$$

where the third equality follows from Beppo Levi's monotone convergence theorem because the supremum in (5.12) can be taken along a sequence of elements $X \in M^\Phi$ such that $\frac{1}{\eta}\frac{d\mathbb{Q}}{d\mathbb{P}}X - H(X)$ is increasing.

It remains to show (5.11). For $\mathrm{T}^\#(\mathbb{Q}) = \infty$, it is an immediate consequence of (5.10). If $\mathrm{T}^\#(\mathbb{Q}) < \infty$, we distinguish four cases:

Case 1: If

$$a := \lim_{x\to\infty} \frac{F(x)}{x} < \infty\,,$$

then $F^*(y) = \infty$ for all $y > a$, and therefore,

$$F(x) = \max_{0\le y\le a} \{xy - F^*(y)\} \quad \text{for } x \ge 0\,.$$

As the supremum of $\sigma(L^{G^*}, M^G)$-continuous functions, the mapping

$$\eta \mapsto \|\eta\|_G^* = \sup_{X\in M^G\,,\, \|X\|_G=1} \mathrm{E}_\mathbb{P}\left[X\,\eta\right]\,,$$

is $\sigma(L^{G^*}, M^G)$-lower semi-continuous on L^{G^*}. Therefore,

$$(X,\eta) \mapsto \left\{\mathrm{E}_\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X - H(X)\,\eta\right] + F^*\left(\|\eta\|_G^*\right)\right\}$$

is a real-valued function on $M^\Phi \times \left\{\eta \in L_+^{G^*} : \|\eta\|_G^* \le a\right\}$ concave in X, convex and $\sigma(L^{G^*}, M^G)$-lower semi-continuous in η. By the Alaoglu–Bourbaki theorem, the convex set $\left\{\eta \in L_+^{G^*} : \|\eta\|_G^* \le a\right\}$ is $\sigma(L^{G^*}, M^G)$-compact. Therefore, we obtain from the Kneser–Fan minimax theorem (Theorem 2 in Fan, 1953) that

$$\begin{aligned}
\mathrm{T}^\#(\mathbb{Q}) &= \sup_{X\in M^\Phi} \left\{\mathrm{E}_\mathbb{Q}\left[X\right] - F\left(\|H(X)\|_G\right)\right\}\\
&= \sup_{X\in M^\Phi} \min_{0\le y\le a\,,\,\eta\in L_+^{G^*}\,,\,\|\eta\|_G^*=1} \left\{\mathrm{E}_\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X\right] - y\mathrm{E}_\mathbb{P}\left[H(X)\,\eta\right] + F^*(y)\right\}\\
&= \sup_{X\in M^\Phi} \min_{\eta\in L_+^{G^*}\,,\,\|\eta\|_G^*\le a} \left\{\mathrm{E}_\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X - H(X)\,\eta\right] + F^*\left(\|\eta\|_G^*\right)\right\}\\
&= \min_{\eta\in L_+^{G^*}\,,\,\|\eta\|_G^*\le a} \sup_{X\in M^\Phi} \left\{\mathrm{E}_\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X - H(X)\,\eta\right] + F^*\left(\|\eta\|_G^*\right)\right\}.
\end{aligned}$$

Together with (5.10), this proves (5.11).

Case 2: If $\lim_{x\to\infty} F(x)/x = \infty$ and $F(x) < \infty$ for all $x \ge 0$, there exists

$$x_0 \ge 1 + \frac{H(0)}{G^{-1}(1)}$$

such that

$$\|\mathbb{Q}\|_\Phi^*\, x_0 - F(x_0) \le \mathrm{T}^\#(\mathbb{Q}) \quad \text{and} \quad F'_+(x_0) \ge \|\mathbb{Q}\|_\Phi^*\,. \tag{5.13}$$

Then,

$$F_\mathbb{Q}(x) := \begin{cases} F(x) & \text{for } 0 \le x \le x_0\\ F(x_0) + F'_+(x_0)(x - x_0) & \text{for } x \ge x_0\end{cases}$$

is an increasing, convex function that is dominated by F and satisfies

$$a_{\mathbb{Q}} := \lim_{x\to\infty} \frac{F_{\mathbb{Q}}(x)}{x} = F'_+(x_0) < \infty\,.$$

For $X \in M^\Phi$ with $\|H(X)\|_G \geq x_0$, we obtain from Lemma 5.1(ii) and (5.13) that

$$\begin{aligned}
& \mathrm{E}_{\mathbb{Q}}[X] - F_{\mathbb{Q}}\left(\|H(X)\|_G\right) \\
\leq\ & \|\mathbb{Q}\|^*_\Phi \left\|X^+\right\|_\Phi - F_{\mathbb{Q}}\left(\|H(X)\|_G\right) \\
\leq\ & \|\mathbb{Q}\|^*_\Phi \|H(X)\|_G - F_{\mathbb{Q}}\left(\|H(X)\|_G\right) \\
=\ & \|\mathbb{Q}\|^*_\Phi\, x_0 - F(x_0) + \left\{\|\mathbb{Q}\|^*_\Phi - F'_+(x_0)\right\}\left\{\|H(X)\|_G - x_0\right\} \\
\leq\ & \|\mathbb{Q}\|^*_\Phi\, x_0 - F(x_0) \leq \mathrm{T}^\#(\mathbb{Q})\,.
\end{aligned}$$

Since $F_{\mathbb{Q}} = F$ on $[0, x_0]$ and $F_{\mathbb{Q}} \leq F$ on (x_0, ∞), this implies

$$\mathrm{T}^\#(\mathbb{Q}) = \sup_{X\in M^\Phi} \left\{\mathrm{E}_{\mathbb{Q}}[X] - F_{\mathbb{Q}}\left(\|H(X)\|_G\right)\right\}\,,$$

and it follows from *Case 1* that

$$\begin{aligned}
\mathrm{T}^\#(\mathbb{Q}) &= \min_{\eta\in L^{G^*}_+,\, \|\eta\|^*_G \leq a_{\mathbb{Q}}}\ \sup_{X\in M^\Phi} \left\{\mathrm{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} X - H(X)\,\eta\right] + F^*_{\mathbb{Q}}\left(\|\eta\|^*_G\right)\right\} \\
&\geq \min_{\eta\in L^{G^*}_+,\, \|\eta\|^*_G \leq a_{\mathbb{Q}}}\ \sup_{X\in M^\Phi} \left\{\mathrm{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} X - H(X)\,\eta\right] + F^*\left(\|\eta\|^*_G\right)\right\}\,,
\end{aligned}$$

which together with (5.10), proves (5.11).

Case 3: If

$$1 + \frac{H(0)}{G^{-1}(1)} \leq z_F := \sup\left\{x \geq 0 : F(x) < \infty\right\} < \infty\,,$$

we set

$$y_0 := F'_-(z_F) \vee \left(\frac{\|\mathbb{Q}\|^*_\Phi\, z_F}{z_F - H(0)/G^{-1}(1)}\right)\,. \tag{5.14}$$

Then

$$F_{\mathbb{Q}}(x) := \begin{cases} F(x) & \text{for } 0 \leq x \leq z_F \\ F(z_F) + y_0\left(x - F(z_F)\right) & \text{for } x \geq z_F \end{cases}$$

is an increasing, convex function that is dominated by F and satisfies

$$a_{\mathbb{Q}} := \lim_{x\to\infty} \frac{F_{\mathbb{Q}}(x)}{x} = y_0 < \infty\,.$$

Let $X \in M^\Phi$ with $\|H(X)\|_G > z_F$. Since $\|H(\lambda X)\|_G$ continuously tends to $\|H(0)\|_G$ for $\lambda \downarrow 0$, it follows from convexity that there exists a $\lambda \in (0,1)$ so that

$$\lambda \left\|(H(X)\right\|_G + (1-\lambda)\left\|H(0)\right\|_G \geq \|H(\lambda X)\|_G = z_F > \frac{H(0)}{G^{-1}(1)} = \|H(0)\|_G\,,$$

and therefore,

$$\|H(X)\|_G \geq \lambda^{-1} z_F - \left(\lambda^{-1} - 1\right)\|H(0)\|_G\,.$$

Now we obtain from Lemma 5.1(ii) that

$$\begin{aligned}
& \mathrm{E}_{\mathbb{Q}}[X] - F_{\mathbb{Q}}(\|H(X)\|_G) \\
= \ & \mathrm{E}_{\mathbb{Q}}[\lambda X] - F_{\mathbb{Q}}(\|H(\lambda X)\|_G) + \mathrm{E}_{\mathbb{Q}}[X - \lambda X] - \{F_{\mathbb{Q}}(\|H(X)\|_G) - F_{\mathbb{Q}}(\|H(\lambda X)\|_G)\} \\
\leq \ & \mathrm{T}^{\#}(\mathbb{Q}) + \left(\lambda^{-1} - 1\right) \mathrm{E}_{\mathbb{Q}}[\lambda X] - y_0 \{\|H(X)\|_G - z_F\} \\
\leq \ & \mathrm{T}^{\#}(\mathbb{Q}) + \left(\lambda^{-1} - 1\right) \|\mathbb{Q}\|_{\Phi}^{*} \left\|(\lambda X)^{+}\right\|_{\Phi} - y_0 \left(\lambda^{-1} - 1\right) (z_F - \|H(0)\|_G) \\
\leq \ & \mathrm{T}^{\#}(\mathbb{Q}) + \left(\lambda^{-1} - 1\right) (\|\mathbb{Q}\|_{\Phi}^{*} \|H(\lambda X)\|_G - y_0 \{z_F - \|H(0)\|_G\}) \\
= \ & \mathrm{T}^{\#}(\mathbb{Q}) + \left(\lambda^{-1} - 1\right) \left(\|\mathbb{Q}\|_{\Phi}^{*} z_F - y_0 \left\{z_F - \frac{H(0)}{G^{-1}(1)}\right\}\right) \\
\leq \ & \mathrm{T}^{\#}(\mathbb{Q})
\end{aligned}$$

by our choice of y_0 (5.14). This shows that

$$\mathrm{T}^{\#}(\mathbb{Q}) = \sup_{X \in M^{\Phi}} \{\mathrm{E}_{\mathbb{Q}}[X] - F_{\mathbb{Q}}(\|H(X)\|_G)\} ,$$

and (5.11) follows as in *Case 2*.

Case 4: Finally, assume

$$z_F = \sup\{x \geq 0 : F(x) < \infty\} < 1 + \frac{H(0)}{G^{-1}(1)} .$$

By (FGH1), there exists $x_0 \in \mathbb{R}$ such that

$$\|H(x_0)\|_G = \frac{H(x_0)}{G^{-1}(1)} < z_F .$$

Hence, there exists a constant $c > 0$ so that

$$1 + \|cH(x_0)\|_G \leq c z_F .$$

Introduce the modified functions

$$\tilde{F}(x) := F(x/c) , \quad x \geq 0 , \quad \text{and} \quad \tilde{H}(x) := cH(x + x_0) , \quad x \in \mathbb{R} .$$

Then, $\tilde{F}, G, \tilde{H}$ still satisfy the conditions (FGH1) and (FGH2). In addition,

$$\sup\left\{x \geq 0 : \tilde{F}(x) < \infty\right\} = c z_F \geq 1 + \|cH(x_0)\|_G = 1 + \frac{\tilde{H}(0)}{G^{-1}(1)} .$$

Hence, it follows from *Case 3* that the minimal penalty function of the shifted risk measure

$$\begin{aligned}
\tilde{\mathrm{T}}(X) &= \min_{s \in \mathbb{R}} \left\{\tilde{F}\left(\left\|\tilde{H}(s - X)\right\|_G\right) - s\right\} \\
&= \min_{s \in \mathbb{R}} \{F(\|H(s + x_0 - X)\|_G) - s\} \\
&= \mathrm{T}(X) + x_0
\end{aligned}$$

is given by

$$
\begin{aligned}
\tilde{\mathrm{T}}^{\#}(\mathbb{Q}) &= \min_{\eta\in L_+^{G^*}} \sup_{X\in M^\Phi} \left\{\mathrm{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X - \tilde{H}(X)\,\eta\right] + \tilde{F}^*\left(\|\eta\|_G^*\right)\right\} \\
&= \min_{\eta\in L_+^{G^*}} \sup_{X\in M^\Phi} \left\{\mathrm{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}(X-x_0) - c\,H(X)\,\eta\right] + F^*\left(\|c\,\eta\|_G^*\right)\right\} \\
&= \min_{\eta\in L_+^{G^*}} \sup_{X\in M^\Phi} \left\{\mathrm{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X - H(X)\,\eta\right] + F^*\left(\|\eta\|_G^*\right)\right\} - x_0\,,
\end{aligned}
$$

where we used the fact that $\tilde{F}^*(y) = F^*(cy)$. This shows that

$$
\begin{aligned}
\mathrm{T}^{\#}(\mathbb{Q}) &= \sup_{X\in M^\Phi}\{\mathrm{E}_{\mathbb{Q}}[-X] - \mathrm{T}(X)\} = \sup_{X\in M^\Phi}\left\{\mathrm{E}_{\mathbb{Q}}[-X] - \tilde{\mathrm{T}}(X)\right\} + x_0 = \tilde{\mathrm{T}}^{\#}(\mathbb{Q}) + x_0 \\
&= \min_{\eta\in L_+^{G^*}} \sup_{X\in M^\Phi} \left\{\mathrm{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}X - H(X)\,\eta\right] + F^*\left(\|\eta\|_G^*\right)\right\}.
\end{aligned}
$$

□

In the next theorem we derive the optimal measure in the robust representation of T under differentiability assumptions. Since Φ, F, G, H are convex, they possess subderivatives φ, f, g, h, that is, φ is a function from $[0,\infty)$ to $\mathbb{R}$ such that

$$
\Phi(z) - \Phi(x) \geq (z-x)\varphi(x) \quad \text{for all } z, x \geq 0\,,
$$

and similarly for F, G and H. The functions φ, f, g, h are in general not unique. But in any case, they are increasing and therefore measurable. If the function F jumps to ∞ at $z_F < \infty$, then f is only defined on the interval $[0, z_F]$. In the proof of Theorem 5.5 we make use of the following relations. They are special instances of Young's inequality.

$$
\begin{array}{rcll}
\Phi(x) \geq xy - \Psi(y) & \text{and} & \Phi(x) = \varphi(x)x - \Psi\circ\varphi(x)\,, & x, y \geq 0 \\
F(x) \geq xy - F^*(y) & \text{and} & F(x) = f(x)x - F^*\circ f(x)\,, & x \in [0, z_F]\,, y \geq 0 \\
G(x) \geq xy - G^*(y) & \text{and} & G(x) = g(x)x - G^*\circ g(x)\,, & x, y \geq 0 \\
H(x) \geq xy - H^*(y) & \text{and} & H(x) = h(x)x - H^*\circ h(x)\,, & x \in \mathbb{R}\,, y \geq 0\,.
\end{array}
$$

We also need the following well-known result. For the convenience of the reader, we provide a short proof.

Lemma 5.4 *Let $X \in M^\Phi \setminus \{0\}$. Then the random variable*

$$
Y := \frac{\mathrm{sign}(X)\,\varphi\left(\frac{|X|}{\|X\|_\Phi}\right)}{\mathrm{E}_{\mathbb{P}}\left[\frac{|X|}{\|X\|_\Phi}\varphi\left(\frac{|X|}{\|X\|_\Phi}\right)\right]}
$$

belongs to L^Ψ and satisfies $\|Y\|_\Phi^* = 1$ as well as $\mathrm{E}_{\mathbb{P}}[XY] = \|X\|_\Phi$.

Proof. Note that $x\varphi(x) \leq \Phi(2x) - \Phi(x) \leq \Phi(2x)$ for all $x \geq 0$. Therefore,

$$\begin{aligned}
& 1 + \mathrm{E}_{\mathbb{P}}\left[\Psi \circ \varphi\left(\frac{|X|}{\|X\|_\Phi}\right)\right] \\
&= \mathrm{E}_{\mathbb{P}}\left[\Phi\left(\frac{|X|}{\|X\|_\Phi}\right)\right] + \mathrm{E}_{\mathbb{P}}\left[\Psi \circ \varphi\left(\frac{|X|}{\|X\|_\Phi}\right)\right] \\
&= \mathrm{E}_{\mathbb{P}}\left[\frac{|X|}{\|X\|_\Phi}\, \varphi\left(\frac{|X|}{\|X\|_\Phi}\right)\right] \leq \mathrm{E}_{\mathbb{P}}\left[\Phi\left(2\frac{|X|}{\|X\|_\Phi}\right)\right] < \infty\,. \qquad (5.15)
\end{aligned}$$

This shows that $\varphi\left(\frac{|X|}{\|X\|_\Phi}\right)$ belongs to L^Ψ. Furthermore, we have

$$\mathrm{E}_{\mathbb{P}}\left[Z\, \varphi\left(\frac{|X|}{\|X\|_\Phi}\right)\right] \leq \mathrm{E}_{\mathbb{P}}\left[\Phi\left(|Z|\right)\right] + \mathrm{E}_{\mathbb{P}}\left[\Psi \circ \varphi\left(\frac{|X|}{\|X\|_\Phi}\right)\right] \leq 1 + \mathrm{E}_{\mathbb{P}}\left[\Psi \circ \varphi\left(\frac{|X|}{\|X\|_\Phi}\right)\right].$$

for all $Z \in L^\Phi$ with $\|Z\|_\Phi \leq 1$. Together with (5.15), this implies

$$\left\|\varphi\left(\frac{|X|}{\|X\|_\Phi}\right)\right\|_\Phi^* = \mathrm{E}_{\mathbb{P}}\left[\frac{|X|}{\|X\|_\Phi}\, \varphi\left(\frac{|X|}{\|X\|_\Phi}\right)\right],$$

and therefore, $\|Y\|_\Phi^* = 1$. $\mathrm{E}_{\mathbb{P}}[XY] = \|X\|_\Phi$ is clear. □

Theorem 5.5 *Assume* (FGH1) *and* (FGH2)*. Let $X \in M^\Phi$ and $s_X \in \mathbb{R}$ such that*

$$\mathrm{T}(X) = F\left(\|H(s_X - X)\|_G\right) - s_X\,.$$

If the mapping

$$s \mapsto F\left(\|H(s - X)\|_G\right)$$

is differentiable at s_X, then

$$\mathrm{T}(X) = E_{\mathbb{Q}_X}[-X] - \mathrm{T}^{\#}(\mathbb{Q}_X)$$

for the probability measure $\mathbb{Q}_X \in \mathcal{D}^\Psi$ given by

$$\frac{d\mathbb{Q}_X}{d\mathbb{P}} = \frac{g\left(\frac{H(s_X - X)}{\|H(s_X - X)\|_G}\right) h(s_X - X)}{\mathrm{E}_{\mathbb{P}}\left[g\left(\frac{H(s_X - X)}{\|H(s_X - X)\|_G}\right) h(s_X - X)\right]}\,.$$

Furthermore,

$$\mathrm{T}^{\#}(\mathbb{Q}_X) = \mathrm{E}_{\mathbb{P}}\left[\eta H^*\left(\frac{1}{\eta}\frac{d\mathbb{Q}_X}{d\mathbb{P}}\right)\right] + F^*\left(\|\eta\|_G^*\right)$$

for

$$\eta = \frac{g\left(\frac{H(s_X - X)}{\|H(s_X - X)\|_G}\right)}{\mathrm{E}_{\mathbb{P}}\left[g\left(\frac{H(s_X - X)}{\|H(s_X - X)\|_G}\right) h(s_X - X)\right]} \in L^{G^*}\,,$$

and

$$\|\eta\|_G^* = f\left(\|H(s_X - X)\|_G\right),$$

where $\frac{1}{\eta}\frac{d\mathbb{Q}_X}{d\mathbb{P}}$ is understood to be 0 on the set $\{d\mathbb{Q}_X/d\mathbb{P} = 0\}$.

Proof. Denote

$$\begin{aligned} x_0 &:= \|H(s_X - X)\|_G \\ Y &:= \frac{g\left(\frac{H(s_X-X)}{\|H(s_X-X)\|_G}\right)}{\mathrm{E}_{\mathbb{P}}\left[\frac{H(s_X-X)}{\|H(s_X-X)\|_G} g\left(\frac{H(s_X-X)}{\|H(s_X-X)\|_G}\right)\right]} \\ \zeta &:= h(s_X - X)\,. \end{aligned}$$

We first show that the random variable $Y\zeta$ belongs to L^Ψ. This follows if we can show that

$$\mathrm{E}_{\mathbb{P}}[Y\zeta Z] < \infty \quad \text{for all } Z \in M^\Phi_+\,. \tag{5.16}$$

Indeed, if (5.16) holds, then $Z \mapsto \mathrm{E}_{\mathbb{P}}[Y\zeta Z]$ is a positive linear functional from M^Φ to $\mathbb{R}$. By Theorem 4.2(i), it must be continuous. Hence, $Y\zeta \in L^\Psi$ since $(L^\Psi, \|.\|^*_\Phi)$ is the norm dual of $(M^\Phi, \|.\|_\Phi)$. To show (5.16), we note that $s_X - X + s x_0 Z$ is in M^Φ for all $s \in \mathbb{R}$. Therefore, it follows from Lemma 5.1(i) that $H(s_X - X + s x_0 Z)/x_0$ is in M^G for all s. Hence, by convexity, we obtain

$$\begin{aligned} & \mathrm{E}_{\mathbb{P}}\left[g\left(\frac{H(s_X - X)}{x_0}\right) h(s_X - X) Z\right] \\ \leq\ & \mathrm{E}_{\mathbb{P}}\left[G\left(\frac{H(s_X - X + x_0 Z)}{x_0}\right) - G\left(\frac{H(s_X - X)}{x_0}\right)\right] < \infty\,, \end{aligned}$$

and (5.16) is proved. Moreover, since $H(s_X - X)$ belongs to M^G, it follows from Lemma 5.4 that

$$Y \text{ is in } L^{G^*},\ \ \|Y\|^*_G = 1,\ \text{and } \mathrm{E}_{\mathbb{P}}[H(s_X - X)\,Y] = \|H(s_X - X)\|_G\,. \tag{5.17}$$

Thus, we get for all $s \in \mathbb{R}$,

$$\begin{aligned} & F\left(\|H(s - X)\|_G\right) - s \qquad (5.18) \\ \geq\ & f(x_0)\ \|H(s - X)\|_G - F^* \circ f(x_0) - s \\ \geq\ & f(x_0)\,\mathrm{E}_{\mathbb{P}}[H(s - X)\,Y] - F^* \circ f(x_0) - s \\ \geq\ & f(x_0)\mathrm{E}_{\mathbb{P}}[\{\zeta(s - X) - H^*(\zeta)\}\,Y] - F^* \circ f(x_0) - s \\ =\ & \mathrm{E}_{\mathbb{P}}[f(x_0)\,Y\,\zeta\,(s - X)] - \mathrm{E}_{\mathbb{P}}[f(x_0)\,Y\,H^*(\zeta)] - F^* \circ f(x_0) - s\,, \qquad (5.19) \end{aligned}$$

with equality for $s = s_X$. (5.19) is an affine function of s which is below the convex function (5.18). For $s = s_X$, the two are equal, and by assumption, (5.18) takes its minimum and is differentiable at s_X. Therefore, (5.19) must be constant in s. This shows that $\frac{d\mathbb{Q}_X}{d\mathbb{P}} := f(x_0)\,Y\,\zeta$ defines a probability measure $\mathbb{Q}_X \in \mathcal{D}^\Psi$. It can be written as

$$\frac{d\mathbb{Q}_X}{d\mathbb{P}} = \frac{g\left(\frac{H(s_X-X)}{\|H(s_X-X)\|_G}\right) h(s_X - X)}{\mathrm{E}_{\mathbb{P}}\left[g\left(\frac{H(s_X-X)}{\|H(s_X-X)\|_G}\right) h(s_X - X)\right]}\,,$$

which implies

$$f(x_0) = \frac{\mathrm{E}_{\mathbb{P}}\left[\frac{H(s_X-X)}{\|H(s_X-X)\|_G} g\left(\frac{H(s_X-X)}{\|H(s_X-X)\|_G}\right)\right]}{\mathrm{E}_{\mathbb{P}}\left[g\left(\frac{H(s_X-X)}{\|H(s_X-X)\|_G}\right) h(s_X - X)\right]}\,.$$

Hence,

$$\eta := \frac{g\left(\frac{H(s_X - X)}{\|H(s_X - X)\|_G}\right)}{\mathrm{E}_{\mathbb{P}}\left[g\left(\frac{H(s_X - X)}{\|H(s_X - X)\|_G}\right) h(s_X - X)\right]} = f(x_0)\, Y\,,$$

and it follows from (5.17) that η is in L^{G^*} with norm $\|\eta\|_G^* = f(x_0)$ and $\frac{1}{\eta}\frac{d\mathbb{Q}_X}{d\mathbb{P}} = \zeta$. By (5.19), we have

$$\mathrm{T}(X) = F\left(\|H(s_X - X)\|_G\right) - s_X = \mathrm{E}_{\mathbb{Q}_X}\left[-X\right] - \mathrm{E}_{\mathbb{P}}\left[\eta\, H^*\left(\frac{1}{\eta}\frac{d\mathbb{Q}_X}{d\mathbb{P}}\right)\right] - F^*\left(\|\eta\|_G^*\right)\,.$$

Thus, it follows from Theorem 5.3 and the definition (4.2) of the minimal penalty function that

$$\mathrm{T}^{\#}(\mathbb{Q}_X) \leq \mathrm{E}_{\mathbb{P}}\left[\eta\, H^*\left(\frac{1}{\eta}\frac{d\mathbb{Q}_X}{d\mathbb{P}}\right)\right] + F^*\left(\|\eta\|_G^*\right) = \mathrm{E}_{\mathbb{Q}_X}\left[-X\right] - \mathrm{T}(X) \leq \mathrm{T}^{\#}(\mathbb{Q}_X)\,,$$

and the proof is complete. □

5.3 Special cases of transformed norm risk measures

5.3.1 $H(x) = x^+$

If the function H is of the form $H(x) = x^+$, then

(5.20) $$\mathrm{T}(X) = \min_{s \in \mathbb{R}} \left\{F\left(\left\|(s - X)^+\right\|_G\right) - s\right\}$$

is a measure of the shortfalls $(s - X)^+$, $s \in \mathbb{R}$, the function Φ equals G, and

$$H^*(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq 1 \\ \infty & \text{if } y > 1 \end{cases}\,.$$

Therefore, it follows from Theorem 5.3 that $\mathrm{T}^{\#}(\mathbb{Q}) = F^*\left(\|\mathbb{Q}\|_G^*\right)$. Moreover, if

$$\mathrm{T}(X) = F\left(\left\|(s_X - X)^+\right\|_G\right) - s_X$$

for some $s_X \in \mathbb{R}$ and $s \mapsto F\left(\|(s - X)^+\|_G\right)$ is differentiable at s_X, then it follows from Theorem 5.5 that

$$\mathrm{T}(X) = \mathrm{E}_{\mathbb{Q}_X}\left[-X\right] - \mathrm{T}^{\#}(\mathbb{Q}_X)$$

for

$$\frac{d\mathbb{Q}_X}{d\mathbb{P}} = \frac{g\left(\frac{(s_X - X)^+}{\|(s_X - X)^+\|_G}\right) 1_{\{X < s_X\}}}{\mathrm{E}_{\mathbb{P}}\left[g\left(\frac{(s_X - X)^+}{\|(s_X - X)^+\|_G}\right) 1_{\{X < s_X\}}\right]}\,,$$

and

$$\mathrm{T}^{\#}(\mathbb{Q}_X) = F^*\left(\|\mathbb{Q}_X\|_G^*\right) = F^* \circ f\left(\left\|(s_X - X)^+\right\|_G\right)\,.$$

If $F(x) = \frac{1}{\alpha}x^\beta$ and $G(x) = x^p$ for parameters (α, β, p) in $(0,1) \times \{1\} \times [1,\infty)$ or $(0,\infty) \times (1,\infty) \times [1,\infty)$, then (5.20) becomes

(5.21) $$\mathrm{T}(X) = \min_{s \in \mathbb{R}} \left\{\frac{1}{\alpha}\left\|(s - X)^+\right\|_p^\beta - s\right\}\,.$$

For $(\alpha, \beta, p) \in (0,1) \times \{1\} \times [1, \infty)$, (5.21) is a coherent risk measure on L^p with minimal penalty function

$$\mathrm{T}^{\#}(\mathbb{Q}) = \begin{cases} 0 & \text{if } \|\mathbb{Q}\|_q \le \frac{1}{\alpha} \\ \infty & \text{if } \|\mathbb{Q}\|_q > \frac{1}{\alpha} \end{cases} , \quad \text{where } q = \frac{p}{p-1} \in (1, \infty] \,.$$

In particular, for $\beta = p = 1$, (5.21) is the so called Conditional-Value-at-Risk or Average-Value-at-Risk; see Rockafellar and Uryasev (2000, 2002) or Föllmer and Schied (2004).

If $(\alpha, \beta, p) \in (0, \infty) \times (1, \infty) \times [1, \infty)$, then (5.21) is a convex monetary risk measure on L^p with

$$\mathrm{T}^{\#}(\mathbb{Q}) = c \, \|\mathbb{Q}\|_q^d \,, \quad \text{where } q = \frac{p}{p-1} \,, \, d = \frac{\beta}{\beta - 1} \,, \, c = \alpha^{d-1} \beta^{1-d} d^{-1} \,.$$

5.3.2 $G(x) = x$:

For $G(x) = x$, we have $\Phi(x) = H(x) - H(0)$, $M^G = L^1$, $L^{G^*} = L^\infty$ and the conditions (FGH1) and (FGH2) reduce to

$$\begin{array}{ll} \text{(FH1)} & F\left(H(x) + \varepsilon\right) < \infty \quad \text{for some } x \in \mathbb{R} \text{ and } \varepsilon > 0 \\ \text{(FH2)} & \lim_{x \to \infty} \left\{F \circ H(x) - x\right\} = \infty \,. \end{array}$$

The corresponding transformed norm risk measure is of the form

$$\mathrm{T}(X) = \min_{s \in \mathbb{R}} \left\{F\left(\mathrm{E}_{\mathbb{P}}\left[H(s - X)\right]\right) - s\right\} \,,$$

and if follows from Theorem 5.3 that

$$\begin{aligned} \mathrm{T}^{\#}(\mathbb{Q}) &= \min_{\eta \in L^\infty_+ , \, \eta \gg \mathbb{Q}} \left\{ \mathrm{E}_{\mathbb{P}}\left[\eta H^*\left(\frac{1}{\eta} \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] + F^*\left(\|\eta\|_\infty\right) \right\} \\ &= \min_{y \in \mathbb{R} , \, y > 0} \left\{ \mathrm{E}_{\mathbb{P}}\left[y H^*\left(\frac{1}{y} \frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] + F^*\left(y\right) \right\} \,, \end{aligned}$$

where the second equality holds because yH is increasing, and $(yH)^* = yH^*\left(\frac{\cdot}{y}\right)$ decreasing in $y > 0$. If

$$\mathrm{T}(X) = F\left(\mathrm{E}_{\mathbb{P}}\left[H(s_X - X)\right]\right) - s_X$$

for some $s_X \in \mathbb{R}$ and the mapping $s \mapsto F\left(\mathrm{E}_{\mathbb{P}}\left[H(s - X)\right]\right)$ is differentiable at s_X, then it follows from Theorem 5.5 that

$$\mathrm{T}(X) = \mathrm{E}_{\mathbb{Q}_X}\left[-X\right] - \mathrm{T}^{\#}(\mathbb{Q}_X)$$

for the measure $\mathbb{Q}_X \in \mathcal{D}^\Psi$ given by

$$\frac{d\mathbb{Q}_X}{d\mathbb{P}} = \frac{h(s_X - X)}{\mathrm{E}_{\mathbb{P}}\left[h(s_X - X)\right]} \,.$$

and

$$\mathrm{T}^{\#}(\mathbb{Q}_X) = \mathrm{E}_{\mathbb{P}}\left[\frac{H^*\left(h(s_X - X)\right)}{\mathrm{E}_{\mathbb{P}}\left[h(s_X - X)\right]}\right] + F^*\left(\frac{1}{\mathrm{E}_{\mathbb{P}}\left[h(s_X - X)\right]}\right) \,.$$

5.4 Transformed loss risk measures

For $F(x) = G(x) = x$, the transformed norm risk measure T reduces to a transformed loss risk measure of the form

$$\mathrm{L}(X) = \min_{s \in \mathbb{R}} \{ \mathrm{E}_{\mathbb{P}} [H(s - X)] - s \} \tag{5.22}$$

on the space M^Φ for $\Phi(x) = H(x) - H(0)$. But now, we can relax the restrictions on H a bit. Instead of requiring it to be non-negative and satisfying (FGH2), we can assume that it is an increasing, convex function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$\lim_{|x| \to \infty} \{ H(x) - x \} = \infty \,. \tag{H}$$

Then, (5.22) is still a real-valued convex monetary risk measure on M^Φ. By (5.2), we obtain for $\mathbb{Q} \in \mathcal{D}^\Psi$,

$$\begin{aligned} \mathrm{L}^{\#}(\mathbb{Q}) &= \sup_{X \in M^\Phi} \left\{ \mathrm{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} X \right] - \mathrm{E}_{\mathbb{P}} [H(X)] \right\} \\ &= \mathrm{E}_{\mathbb{P}} \left[\sup_{X \in M^\Phi} \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} X - H(X) \right\} \right] = \mathrm{E}_{\mathbb{P}} \left[H^* \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] . \end{aligned} \tag{5.23}$$

The second equality follows from Beppo Levi's monotone convergence theorem because the supremum can be taken along a sequence of X such that $\frac{d\mathbb{Q}}{d\mathbb{P}} X - H(X)$ is increasing.

If $X \in M^\Phi$ and $s_X \in \mathbb{R}$ are such that $\mathrm{L}(X) = \mathrm{E}_{\mathbb{P}} [H(s_X - X)] - s_X$ and the mapping $s \mapsto \mathrm{E}_{\mathbb{P}} [H(s - X)]$ is differentiable at s_X, a simplified version of the proof of Theorem 5.5 yields that

$$\frac{d\mathbb{Q}_X}{d\mathbb{P}} = h(s_X - X)$$

defines a probability measure $\mathbb{Q}_X$ in $\mathcal{D}^\Psi$ which satisfies

$$\mathrm{L}(X) = \mathrm{E}_{\mathbb{Q}_X} [-X] - \mathrm{L}^{\#}(\mathbb{Q}_X) \,,$$

and

$$\mathrm{L}^{\#}(\mathbb{Q}_X) = \mathrm{E}_{\mathbb{P}} [H^* (h(s_X - X))] \,.$$

If $H^*(1) = 0$, the last expression in (5.23) is an f-divergence after Csiszar (1967) and can be interpreted as a distance between $\mathbb{Q}$ and $\mathbb{P}$. Duality results similar to (5.23) have been proved in Ben-Tal and Teboulle (1987) and Schied (2007).

If, for example,

$$H(x) = \frac{1}{\theta} \exp(\theta x - 1) \quad \text{for a positive parameter } \theta \,,$$

then

$$H^*(y) = \frac{1}{\theta} y \log(y) \,,$$

and (5.22) is the entropic risk measure

$$\rho(X) = \frac{1}{\theta} \log \left(\mathrm{E}_{\mathbb{P}} [\exp(-\theta X)] \right) \,, \quad X \in M^\Phi \,,$$

whose minimal penalty function equals the relative entropy

$$\rho^{\#}(\mathbb{Q}) = \frac{1}{\theta}\mathrm{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \quad \mathbb{Q}\in\mathcal{D}^{\Psi}\,.$$

If

$$H(x) = \frac{\theta}{2}(x^+)^2 + \frac{1}{2\theta} \quad \text{for a positive parameter } \theta\,,$$

then

$$H^*(y) = \frac{1}{2\theta}(y^2-1)\,, \quad y\geq 0\,,$$

and (5.22) is the negative of the monotone hull of a mean-variance preference functional

$$\rho(X) = \min_{s\in\mathbb{R}}\left\{\frac{\theta}{2}\mathrm{E}_{\mathbb{P}}\left[\left((s-X)^+\right)^2\right] - s\right\} + \frac{1}{2\theta}\,,$$

whose minimal penalty function is the Gini index

$$\rho^{\#}(\mathbb{Q}) = \frac{1}{2\theta}\mathrm{E}_{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2 - 1\right] = \frac{1}{2\theta}\mathrm{E}_{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}} - 1\right)^2\right]\,;$$

see, for instance, Maccheroni et al. (2007).

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