Arbitrage in fractional Brownian motion models

Patrick Cheridito ⋆

Departement für Mathematik, Eidgenössische Technische Hochschule Zürich, 8092 Zürich, Switzerland (e-mail: dito@math.ethz.ch)

October 2002

Abstract We construct arbitrage strategies for a financial market that consists of a money market account and a stock whose discounted price follows a fractional Brownian motion with drift or an exponential fractional Brownian motion with drift. Then we show how arbitrage can be excluded from these models by restricting the class of trading strategies.

Key words: fractional Brownian motion, arbitrage, strong arbitrage, exclusion of arbitrage

JEL Classification: C60, G10

Mathematics Subject Classification (2000): 60G15, 60K30, 91B28

1 Introduction

We consider a market that consists of a money market account and a stock that pays no dividends. All economic activity takes place in a time interval $[0, T]$ for some $T \in (0, \infty)$. Borrowing and short-selling are allowed, the borrowing rate is equal to the lending rate, and it is possible to buy and sell any fraction of stock shares. Moreover, there exist no transaction costs and stock shares can be be bought and sold at the same price. We assume that there exist two stochastic processes $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ on a probability space $(\Omega, \mathcal{A}, P)$ such that money in the money market account evolves according to $(X_t)_{t\in[0,T]}$ and the stock price follows $(Y_t)_{t\in[0,T]}$.

⋆ This paper is part of the author's doctoral dissertation written under the supervision of Freddy Delbaen. Financial support from Credit Suisse is gratefully acknowledged.

A fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1]$, is a continuous, centered Gaussian process $(B_t^H)_{t \in \mathbb{R}}$ with covariance

$$\operatorname{Cov}\left(B_t^H, B_s^H\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \; t, s \in \mathbb{R}\,. \tag{1.1}$$

$B^{\frac{1}{2}}$ is a two-sided Brownian motion. The paths of B^1 are straight lines with a normally distributed slope. For $H \in (\frac{1}{2}, 1]$ the correlation of two increments of B^H over non-interlapping time-intervals is positive, and for $H \in (0, \frac{1}{2})$ it is negative. It can easily be seen from (1.1) that $\mathrm{E}\left[(B_t^H - B_s^H)^2\right] = |t-s|^{2H}$. Hence, Kolmogorov's continuity criterion applies (see e.g. Theorem I.2.1 in Revuz and Yor (1999)), and it follows that almost all paths of B^H are locally Hölder continuous of order α for every $\alpha < H$. Furthermore, B^H has stationary increments and is H-self-similar, that is, for all $a > 0$, $(B_{at}^H)_{t \in \mathbb{R}}$ has the same distribution as $(a^H B_t^H)_{t \in \mathbb{R}}$. More details about fBm can be found in Section 7.2 of Samorodnitsky and Taqqu (1994). We call

$$X_t = 1, \quad Y_t = Y_0 + \nu t + \sigma B_t^H \,, \; t \in [0,T]\,, \tag{1.2}$$

the fractional Bachelier model and

$$X_t = \exp(rt), \quad Y_t = Y_0 \exp\left(\{r + \nu\}\, t + \sigma B_t^H\right) \,, \; t \in [0,T]\,, \tag{1.3}$$

the fractional Samuelson model or fractional Black-Scholes model. For the moment we assume that Y_0, σ are positive constants and ν, r are real constants, but all our results will also hold true if νt is replaced by any function in $C^1(\mathbb{R}_+)$ and rt by an arbitrary right-continuous stochastic process. For a discussion of the empirical evidence of correlation in stock price returns see e.g Cutland et al. (1995) or Willinger et al. (1999) and the references therein.

For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $(B_t^H)_{t \geq 0}$ is not a semimartingale (see e.g. Liptser and Shiryaev (1989) or Rogers (1997)). Hence, it immediately follows from Theorem 7.2 of Delbaen and Schachermayer (1994) that in both models (1.2) and (1.3) there exists a weak form of arbitrage called 'free lunch with vanishing risk' consisting of simple integrands that are predictable with respect to the smallest filtration that satisfies the usual assumptions and contains the filtration generated by the discounted stock price process. Rogers (1997), Shiryaev (1998) and Salopek (1998) give arbitrage strategies for fBm models.

Rogers (1997) constructs arbitrage for the fractional Bachelier model (1.2). His strategy consists of a combination of buy and hold strategies and works for all Hurst parameters $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. However, as self-similarity of the process Y is essential for its construction, Rogers' arbitrage only exists in the case $\nu = 0$, i.e. $Y_t = Y_0 + \sigma B_t^H$. Moreover, Rogers models Y_t for $t \in (-\infty, 0]$ and to generate a profit on the time interval $[-1, 0)$, his arbitrage strategy needs to know the whole history of Y from time $-\infty$ until the present.

In Shiryaev (1998) only the case $H \in (\frac{1}{2}, 1)$ is treated. An integral with respect to B^H is defined and it is indicated how it can be shown that for regular enough functions F, the modified Itô formula

$$dF(t, B_t^H) = \partial_1 F(t, B_t^H)dt + \partial_2 F(t, B_t^H)dB_t^H \tag{1.4}$$

holds. Using this for the fractional Bachelier model (1.2) with $H \in (\frac{1}{2}, 1)$, one can choose a $c > 0$ and set

$$\vartheta_t^0 := -c\left(\nu t + \sigma B_t^H\right)^2 - 2cY_0\left(\nu t + \sigma B_t^H\right), \quad \vartheta_t^1 := 2c\left(\nu t + \sigma B_t^H\right)$$

to obtain

$$\vartheta_t^0 X_t + \vartheta_t^1 Y_t = \vartheta_0^0 X_0 + \vartheta_0^1 Y_0 + \int_0^t \vartheta_u^1 dY_u = c\left(\nu t + \sigma B_t^H\right)^2 .$$

Hence, if continuous adjustment of the portfolio is allowed, $(\vartheta^0, \vartheta^1)$ is a self-financing arbitrage strategy for the fractional Bachelier model. For the fractional Samuelson model (1.3) with $H \in (\frac{1}{2}, 1)$, one can set for all $c > 0$,

$$\vartheta_t^0 := cY_0\left\{1 - \exp\left(2\nu t + 2\sigma B_t^H\right)\right\}, \quad \vartheta_t^1 := 2c\left\{\exp\left(\nu t + \sigma B_t^H\right) - 1\right\} .$$

It follows from (1.4) that

$$\begin{aligned} \vartheta_t^0 X_t + \vartheta_t^1 Y_t &= \vartheta_0^0 X_0 + \vartheta_0^1 Y_0 + \int_0^t \vartheta_u^0 dX_u + \int_0^t \vartheta_u^1 dY_u \\ &= cY_0 \exp(rt)\left\{\exp\left(\nu t + \sigma B_t^H\right) - 1\right\}^2 , \end{aligned}$$

which shows that $(\vartheta^0, \vartheta^1)$ is a self-financing arbitrage strategy for the fractional Samuelson model.

More generally, it follows from Young's theorem on Stieltjes integrability (see Young (1936)) that if a stochastic process $(Y_t)_{t \geq 0}$ is almost surely continuous and of bounded p-variation for some $p < 2$ (this is the case for Y in (1.2) and (1.3) when $H \in (\frac{1}{2}, 1)$), then for a real function f on $\mathbb{R}$ that is locally Lipschitz, the path-wise Riemann-Stieltjes integral $\int_0^t f(Y_u)dY_u$ exists for all $t \geq 0$ and equals $F(Y_t) - F(Y_0)$, where $F' = f$. In Salopek (1998) this is used to construct a self-financing arbitrage strategy for two financial assets whose price processes X and Y are almost surely continuous, of bounded p-variation for some $p < 2$ and such that $X_t \neq Y_t$ for all t.

In this paper we want to do the following two things: for a class of models that contains (1.2) and (1.3) for all $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, first, construct as simple as possible arbitrage strategies and secondly, find an as big as possible class of trading strategies that does not contain arbitrage.

The structure of the paper is as follows: In Section 2 we introduce different classes of trading strategies and define the notions of 'free lunch with vanishing risk', 'arbitrage' and 'strong arbitrage'. In Section 3 we construct arbitrage strategies. As in Rogers (1997) our arbitrage strategies consist of combinations of buy and hold strategies. Therefore we need no integration theory for fBm. Moreover, to generate a profit on the time interval $[0, T]$, our strategies need only know the history of the discounted stock price Y/X from time 0 on. However, these strategies can only be performed if it is possible to buy and sell within arbitrarily small time intervals. In Section 4 we show that arbitrage can be ruled out from models of the form (1.2) and (1.3) by introducing a minimal amount of time $h > 0$ that must lie between two consecutive transactions.

2 Trading strategies

In this section the time interval is an arbitrary compact interval $[a, b]$. Money can be invested in a money market account where money grows according to a stochastic process $(X_t)_{t\in[a,b]}$ and a stock whose price follows a stochastic process $(Y_t)_{t\in[a,b]}$. We first give the definition of different notions of arbitrage and specify the trading strategies afterwards. For the time being a trading strategy is just a pair $\vartheta = (\vartheta^0, \vartheta^1)$ of stochastic processes $(\vartheta^0_t)_{t\in[a,b]}$ and $(\vartheta^1_t)_{t\in[a,b]}$. $\vartheta^0_t X_t$ describes the money in the money market account at time t and ϑ^1_t the number of stock shares held at time t. Hence, the evolution of the portfolio value of a strategy ϑ is given by

$$V^\vartheta_t := \vartheta^0_t X_t + \vartheta^1_t Y_t \,,\; t \in [a, b] \,.$$

Since we want to use X as numéraire, we require it to be positive. We set

$$\tilde{Y}_t := \frac{Y_t}{X_t} \quad \text{and} \quad \tilde{V}^\vartheta_t := \frac{V^\vartheta_t}{X_t} \,, t \in [a, b] \,.$$

Definition 2.1 *Let ξ be a $[0, \infty]$-valued random variable with $P[\xi > 0] > 0$.*

a) *A sequence of trading strategies $\{\vartheta(n)\}_{n=1}^\infty$ is a ξ-FLVR (ξ-free lunch with vanishing risk) if*

$$\lim_{n\to\infty} \left(\tilde{V}^{\vartheta(n)}_b - \tilde{V}^{\vartheta(n)}_a\right) = \xi \quad \textit{in probability, and}$$
$$\lim_{n\to\infty} \left\| \left(\tilde{V}^{\vartheta(n)}_b - \tilde{V}^{\vartheta(n)}_a\right)^- \right\|_\infty = 0 \,.$$

$\{\vartheta(n)\}_{n=1}^\infty$ is a FLVR if it is a ξ'-FLVR for some $[0, \infty]$-valued random variable ξ' with $P[\xi' > 0] > 0$.

b) *A trading strategy ϑ is a ξ-arbitrage if $\tilde{V}^\vartheta_b - \tilde{V}^\vartheta_a = \xi$ almost surely. ϑ is an arbitrage if it is a ξ'-arbitrage for some $[0, \infty]$-valued random variable ξ' with $P[\xi' > 0] > 0$.*

c) *A trading strategy ϑ is a strong arbitrage if there exists a constant $c > 0$ such that $\tilde{V}^\vartheta_b - \tilde{V}^\vartheta_a \geq c$ almost surely.*

It is clear that we must put certain restrictions on a trading strategy to give it an economic meaning. First of all, trading strategies should only be based on available information. To describe the evolution of information we introduce a family of σ-algebras $\mathbb{F} = (\mathcal{F}_t)_{t\in[a,b]}$. We assume that at any time $t \in [a,b]$, X_t and Y_t can be observed and no information is lost over time. In other words, $\mathbb{F}$ is a filtration and

$$\mathcal{F}_t^{X,Y} := \sigma\left((X_u)_{u\in[0,t]}, (Y_u)_{u\in[0,t]} \right) \subset \mathcal{F}_t \quad \text{for all } t \in [a,b]\,.$$

Note that

$$\mathcal{F}_t^{\tilde{Y}} := \sigma\left(\left(\tilde{Y}_u\right)_{u\in[0,t]} \right) \subset \mathcal{F}_t^{X,Y} \quad \text{for all } t \in [a,b]\,.$$

Furthermore, we require X and Y to be progressively measurable with respect to $\mathbb{F}$. This is in particular the case when X and Y are right-continuous, and it ensures that for all $\mathbb{F}$-stopping times τ, the stopped processes $(X_{\tau\wedge t})_{t\in[a,b]}$ and $(Y_{\tau\wedge t})_{t\in[a,b]}$ are also progressively measurable with respect to $\mathbb{F}$. To construct arbitrage in fBm models of the form (1.2) or (1.3) it is enough to consider combinations of buy and hold strategies.

Definition 2.2

a) *The set of simple predictable integrands is given by*
$\mathbf{S}(\mathbb{F}) := \{g_0 1_{\{a\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j,\tau_{j+1}]} \,:\, n \geq 2,\ a = \tau_1 \leq \ldots \leq \tau_n = b;$
all τ_j*'s are* $\mathbb{F}$*-stopping times;* g_0 *is a real,* $\mathcal{F}_a$*-measurable random variable; and the other* g_j*'s are real,* $\mathcal{F}_{\tau_j}$*-measurable random variables*$\}$.
The class of simple predictable trading strategies is given by
$\Theta^{\mathbf{S}}(\mathbb{F}) := \left\{\vartheta = \left(\vartheta^0, \vartheta^1\right) \,:\, \vartheta^0, \vartheta^1 \in \mathbf{S}(\mathbb{F})\right\}$.

b) *The set of almost simple predictable integrands is given by*
$\mathbf{aS}(\mathbb{F}) := \{g_0 1_{\{a\}} + \sum_{j=1}^{\infty} g_j 1_{(\tau_j,\tau_{j+1}]} \,:\, a = \tau_1 \leq \tau_2 \leq \ldots \leq b;$
all τ_j*'s are* $\mathbb{F}$*-stopping times;* g_0 *is a real,* $\mathcal{F}_a$*-measurable random variable; the other* g_j*'s are real,* $\mathcal{F}_{\tau_j}$*-measurable random variables;* $P\,[\exists\, j \text{ such that } \tau_j = b] = 1\}$.
The class of almost simple predictable trading strategies is given by
$\Theta^{\mathbf{aS}}(\mathbb{F}) := \left\{\vartheta = \left(\vartheta^0, \vartheta^1\right) \,:\, \vartheta^0, \vartheta^1 \in \mathbf{aS}(\mathbb{F})\right\}$.

c) *For* $\vartheta^1 = g_0 1_{\{a\}} + \sum_{j=1}^{\infty} g_j 1_{(\tau_j,\tau_{j+1}]} \in \mathbf{aS}(\mathbb{F})$ *we define*
$\left(\vartheta^1 \cdot Y\right)_t := \sum_{j=1}^{\infty} g_j (Y_{\tau_{j+1}\wedge t} - Y_{\tau_j \wedge t}),\ t \in [a,b]$.
(Note that this is almost surely a sum of finitely many terms and the process $(\left(\vartheta^1 \cdot Y\right)_t)_{t\in[a,b]}$ *is progressively measurable because* $(Y_t)_{t\in[a,b]}$ *is.)*

Definition 2.3 *Let* $\vartheta = (\vartheta^0, \vartheta^1) \in \Theta^{\mathbf{aS}}(\mathbb{F})$*. There exist stopping times*

$$a = \tau_1 \leq \tau_2 \leq \ldots \leq b$$

such that ϑ^0 *and* ϑ^1 *can be written in the form*

$$\vartheta^0 = f_0 1_{\{a\}} + \sum_{j=1}^{\infty} f_j 1_{(\tau_j,\tau_{j+1}]}, \quad \vartheta^1 = g_0 1_{\{a\}} + \sum_{j=1}^{\infty} g_j 1_{(\tau_j,\tau_{j+1}]}\,. \tag{2.1}$$

We set $\tau_0 = a - 1$ *and call* ϑ *self-financing for* (X, Y) *if for all* $j \geq 1$, $k = 1, \dots, j$ *and* $l \geq 0$,

$$1_{\{\tau_{j-k} < \tau_{j-k+1} = \tau_{j+l} < \tau_{j+l+1}\}} \left\{ (f_{j+l} - f_{j-k}) X_{\tau_j} + (g_{j+l} - g_{j-k}) Y_{\tau_j} \right\} \overset{\text{a.s.}}{=} 0 . \tag{2.2}$$

(Note that property (2.2) is independent of the representation (2.1) of ϑ*.) Furthermore, we set*

$$\Theta^{\mathbf{S}}_{\text{sf}}(\mathbb{F}) := \left\{ \vartheta \in \Theta^{\mathbf{S}}(\mathbb{F}) : \vartheta \textit{ is self-financing} \right\} \quad \textit{and}$$
$$\Theta^{\mathbf{aS}}_{\text{sf}}(\mathbb{F}) := \left\{ \vartheta \in \Theta^{\mathbf{aS}}(\mathbb{F}) : \vartheta \textit{ is self-financing} \right\} .$$

Proposition 2.4 *Let* $\vartheta = (\vartheta^0, \vartheta^1) \in \Theta^{\mathbf{aS}}(\mathbb{F})$*. Then the following are equivalent:*

(i) ϑ *is self-financing for* (X, Y)
(ii) *almost surely,* $V^\vartheta_t = V^\vartheta_a + \left(\vartheta^0 \cdot X\right)_t + \left(\vartheta^1 \cdot Y\right)_t$ *for all* $t \in [a, b]$
(iii) ϑ *is self-financing for* $(1, \tilde{Y})$
(iv) *almost surely,* $\tilde{V}^\vartheta_t = \tilde{V}^\vartheta_a + \left(\vartheta^1 \cdot \tilde{Y}\right)_t$ *for all* $t \in [a, b]$

Proof Let $a = \tau_1 \leq \tau_2 \leq \ldots \leq b$ be an increasing sequence of $\mathbb{F}$-stopping times such that

$$\vartheta^0 = f_0 1_{\{a\}} + \sum_{j=1}^\infty f_j 1_{(\tau_j, \tau_{j+1}]}, \quad \vartheta^1 = g_0 1_{\{a\}} + \sum_{j=1}^\infty g_j 1_{(\tau_j, \tau_{j+1}]} .$$

(i) $\Rightarrow$ (ii): It follows from (i) that there exists a measurable $\Omega' \subset \Omega$ with $P[\Omega'] = 1$ such that for each $\omega \in \Omega'$, equation (2.2) holds for all $j \geq 1$, $k = 1, \dots, j$ and $l \geq 0$ simultaneously. For $t = a$, the equation in (ii) holds for all $\omega \in \Omega$. Furthermore, there exists a measurable $\Omega'' \subset \Omega'$ with $P[\Omega''] = 1$ such that for all $\omega \in \Omega''$, there exists for every $t \in (a, b]$, a $j \in \mathbb{N}$, such that $t \in (\tau_j, \tau_{j+1}]$, and

$$\begin{aligned} & V^\vartheta_a + \left(\vartheta^0 \cdot X\right)_t + \left(\vartheta^1 \cdot Y\right)_t = f_0 X_{\tau_1} + g_0 Y_{\tau_1} + \sum_{i=1}^{j-1} f_i \left(X_{\tau_{i+1}} - X_{\tau_i}\right) \\ & + f_j \left(X_t - X_{\tau_j}\right) + \sum_{i=1}^{j-1} g_i \left(Y_{\tau_{i+1}} - Y_{\tau_i}\right) + g_j \left(Y_t - Y_{\tau_j}\right) \\ = & \sum_{i=1}^{j} X_{\tau_i} (f_{i-1} - f_i) + \sum_{i=1}^{j} Y_{\tau_i} (g_{i-1} - g_i) + f_j X_t + g_j Y_t = \vartheta^0_t X_t + \vartheta^1_t Y_t . \end{aligned}$$

This shows (ii).
(ii) $\Rightarrow$ (i): Let $j \geq 1$, $k = 1, \dots, j$ and $l \geq 0$.
On $\{\tau_{j-k} < \tau_{j-k+1} = \tau_{j+l} < \tau_{j+l+1}\}$ we have

$$\begin{aligned} & (f_{j+l} - f_{j-k}) X_{\tau_j} + (g_{j+l} - g_{j-k}) Y_{\tau_j} \\ = & \left(f_{j+l} X_{\tau_{j+l+1}} + g_{j+l} Y_{\tau_{j+l+1}}\right) - \left(f_{j-k} X_{\tau_j} + g_{j-k} Y_{\tau_j}\right) \end{aligned}$$

$$
\begin{aligned}
&-f_{j+l}\left(X_{\tau_{j+l+1}}-X_{\tau_j}\right)-g_{j+l}\left(Y_{\tau_{j+l+1}}-Y_{\tau_j}\right)\\
&=\left(\vartheta^0_{\tau_{j+l+1}}X_{\tau_{j+l+1}}+\vartheta^1_{\tau_{j+l+1}}Y_{\tau_{j+l+1}}\right)-\left(\vartheta^0_{\tau_j}X_{\tau_j}+\vartheta^1_{\tau_j}Y_{\tau_j}\right)\\
&\quad-\left[\vartheta^0_aX_a+\vartheta^1_aY_a+\sum_{i=1}^{j+l}f_i\left(X_{\tau_{i+1}}-X_{\tau_i}\right)+\sum_{i=1}^{j+l}g_i\left(Y_{\tau_{i+1}}-Y_{\tau_i}\right)\right]\\
&\quad+\left[\vartheta^0_aX_a+\vartheta^1_aY_a+\sum_{i=1}^{j-1}f_i\left(X_{\tau_{i+1}}-X_{\tau_i}\right)+\sum_{i=1}^{j-1}g_i\left(Y_{\tau_{i+1}}-Y_{\tau_i}\right)\right]\overset{\text{a.s.}}{=}0\,,
\end{aligned}
$$

which proves (i).
The equivalence of (i) and (iii) is trivial, and the equivalence of (iii) and (iv) can be shown in the same way as the equivalence of (i) and (ii). □

Remark 2.5 It follows from Proposition 2.4 that for all $\vartheta \in \Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F})$, almost surely,

$$
\vartheta^0_t = \tilde{V}^{\vartheta}_a + \left(\vartheta^1 \cdot \tilde{Y}\right)_t - \vartheta^1_t \tilde{Y}_t\,,\; t \in [a,b]\,.
$$

This shows that if we identify indistinguishable processes, the map

$$
\vartheta = \left(\vartheta^0, \vartheta^1\right) \mapsto \left(\tilde{V}^{\vartheta}_a, \vartheta^1\right)
$$

is a bijection from $\Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F})$ to $L^0(\mathcal{F}_a) \times \mathbf{aS}(\mathbb{F})$. In particular, there exists for all $(\xi, \vartheta^1) \in L^0(\mathcal{F}_a) \times \mathbf{aS}(\mathbb{F})$, a unique $\vartheta^0 \in \mathbf{aS}(\mathbb{F})$ such that $\vartheta = (\vartheta^0, \vartheta^1)$ is in $\Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F})$ and $\tilde{V}^{\vartheta}_a = \xi$.

In $\Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F}^{\tilde{Y}})$ there exist so called doubling strategies which can create strong arbitrage even in the standard Samuelson (or Black-Scholes) model ((1.3) with $H = \frac{1}{2}$). It was noticed by Harrison and Pliska (1981) that they can be ruled out by putting an admissibility condition on the trading strategies. We use the admissibility condition of Delbaen and Schachermayer (1994). It is more liberal than the one of Harrison and Pliska (1981) but restrictive enough to exclude arbitrage in the standard Samuelson model.

Definition 2.6 *For $c \geq 0$, we call $\vartheta \in \Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F})$ c-admissible if*

$$
\textit{almost surely,}\quad \inf_{t\in[a,b]}\left(\tilde{V}^{\vartheta}_t - \tilde{V}^{\vartheta}_a\right) = \inf_{t\in[a,b]}\left(\vartheta^1 \cdot \tilde{Y}\right)_t \geq -c\,.
$$

We call ϑ admissible if it is c-admissible for some $c \geq 0$. Furthermore, we set

$$
\begin{aligned}
&\Theta^{\mathbf{S}}_{\mathrm{sf,adm}}(\mathbb{F}) := \left\{\vartheta \in \Theta^{\mathbf{S}}_{\mathrm{sf}}(\mathbb{F}) \,:\, \vartheta \textit{ is admissible}\right\} \quad \textit{and}\\
&\Theta^{\mathbf{aS}}_{\mathrm{sf,adm}}(\mathbb{F}) := \left\{\vartheta \in \Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F}) \,:\, \vartheta \textit{ is admissible}\right\}.
\end{aligned}
$$

3 Construction of arbitrage

Theorem 3.1 *Let B^H be a fBm. Let $T \in (0,\infty)$, $\nu \in C^1[0,T]$ and $\sigma > 0$. Then in all four cases*

$$\begin{array}{lll} \text{(i)} & H \in (\frac{1}{2},1), & \tilde{Y}_t = \nu(t) + \sigma B^H_t \,,\; t \in [0,T] \\ \text{(ii)} & H \in (\frac{1}{2},1), & \tilde{Y}_t = \exp\left(\nu(t) + \sigma B^H_t\right) ,\; t \in [0,T] \\ \text{(iii)} & H \in (0,\frac{1}{2}), & \tilde{Y}_t = \nu(t) + \sigma B^H_t \,,\; t \in [0,T] \\ \text{(iv)} & H \in (0,\frac{1}{2}), & \tilde{Y}_t = \exp\left(\nu(t) + \sigma B^H_t\right) ,\; t \in [0,T] \,, \end{array}$$

there exists for every constant $c > 0$ and all $n \in \mathbb{N}$, a $\vartheta^1(n) \in \mathbf{S}(\mathbb{F}^{\tilde{Y}})$ such that

$$\begin{array}{l} \text{a) } P\left[\left(\vartheta^1(n) \cdot \tilde{Y}\right)_T = c\right] > 1 - \frac{1}{n} \quad \textit{and} \\ \text{b) } \inf_{t\in[0,T]} \left(\vartheta^1(n) \cdot \tilde{Y}\right)_t \geq -\frac{1}{n} \,. \end{array}$$

In particular, the strategies $\vartheta(n) = (\vartheta^0(n), \vartheta^1(n)) \in \Theta^{\mathbf{S}}_{\mathrm{sf,adm}}(\mathbb{F}^{\tilde{Y}})$, $n \in \mathbb{N}$, where $\vartheta^0(n)$ is given by

$$\vartheta^0_t(n) = \left(\vartheta^1(n) \cdot \tilde{Y}\right)_t - \vartheta^1_t(n)\tilde{Y}_t \,,\; t \in [0,T] \,,\; n \in \mathbb{N} \,,$$

form a c-FLVR. In the cases (iii) *and* (iv)*, $\vartheta^1(n)$ can be chosen such that also*

$$\text{c) } \left|\vartheta^1(n)\right| \leq \frac{1}{n} \,.$$

Theorem 3.2 *In all four cases* (i)-(iv) *of Theorem 3.1 there exists for every constant $c > 0$, a $\frac{1}{c}$-admissible c-arbitrage $\vartheta \in \Theta^{\mathbf{aS}}_{\mathrm{sf,adm}}(\mathbb{F}^{\tilde{Y}})$. In the cases* (iii) *and* (iv)*, ϑ can be chosen such that $\left|\vartheta^1\right| \leq \frac{1}{c}$.*

For the proofs of Theorems 3.1 and 3.2 we need the following three lemmas.

Lemma 3.3 *Let $(Z_t)_{t\in[a,b]}$ be a continuous stochastic process. If*

$$P\left[Z_b = Z_a\right] = 0 \,, \tag{3.1}$$

and for all $\varepsilon > 0$, there exist $\mathbb{F}^Z$-stopping times $a = \tau_0 \leq \ldots \leq \tau_n = b$ such that

$$P\left[\max_{t\in[a,b]} \sum_{j=0}^{n-1} \left(Z_{\tau_{j+1}\wedge t} - Z_{\tau_j \wedge t}\right)^2 \geq \varepsilon\right] < \varepsilon, \tag{3.2}$$

then there exists for all $M > 0$, a $\beta \in \mathbf{S}(\mathbb{F}^Z)$ such that

$$\begin{array}{l} \text{a) } P\left[(\beta \cdot Z)_b < M\right] < \frac{1}{M} \quad \textit{and} \\ \text{b) } \inf_{t\in[a,b]} (\beta \cdot Z)_t \geq -\frac{1}{M}. \end{array}$$

Proof Let $M > 0$. It follows from (3.1) and (3.2) that there exist an $\varepsilon > 0$ such that

$$P\left[(Z_b - Z_a)^2 < \varepsilon\right] < \frac{1}{2M} \tag{3.3}$$

and $\mathbb{F}^Z$-stopping times $a = \tau_0 \leq \ldots \leq \tau_n = b$, such that

$$P\left[\max_{t \in [a,b]} \sum_{j=0}^{n-1} \left(Z_{\tau_{j+1} \wedge t} - Z_{\tau_j \wedge t}\right)^2 \geq \frac{\varepsilon}{M^2+1}\right] < \frac{1}{2M}\,. \tag{3.4}$$

Since Z is continuous,

$$\xi := \inf\left\{t \in [a,b] \,:\, \sum_{j=0}^{n-1} \left(Z_{\tau_{j+1} \wedge t} - Z_{\tau_j \wedge t}\right)^2 \geq \frac{\varepsilon}{M^2+1}\right\} \quad (\text{set } \inf \emptyset = b) \tag{3.5}$$

is an $\mathbb{F}^Z$-stopping time (see e.g. Proposition I.4.5 in Revuz and Yor (1999)), and (3.4) implies

$$P\left[\xi < b\right] < \frac{1}{2M}\,. \tag{3.6}$$

Furthermore,

$$\beta := \frac{2}{\varepsilon}\left(M + \frac{1}{M}\right) \sum_{j=0}^{n-1} \left(Z_{\tau_j} - Z_a\right) 1_{(\tau_j, \tau_{j+1}]} 1_{[0,\xi]}$$

is in $\mathbf{S}(\mathbb{F}^Z)$, and for all $t \in [a,b]$,

$$(\beta \cdot Z)_t = \frac{M + \frac{1}{M}}{\varepsilon}\left[(Z_{t \wedge \xi} - Z_a)^2 - \sum_{j=0}^{n-1} \left(Z_{\tau_{j+1} \wedge t \wedge \xi} - Z_{\tau_j \wedge t \wedge \xi}\right)^2\right]. \tag{3.7}$$

This together with (3.5) implies b). From (3.7), (3.6) and (3.3) it follows that

$$\begin{aligned}
&P\left[(\beta \cdot Z)_b < M\right] \\
&= P\left[\frac{M + \frac{1}{M}}{\varepsilon}\left\{(Z_\xi - Z_a)^2 - \sum_{j=0}^{n-1} \left(Z_{\tau_{j+1} \wedge \xi} - Z_{\tau_j \wedge \xi}\right)^2\right\} < M\right] \\
&\leq P\left[(Z_\xi - Z_a)^2 < \varepsilon\right] \leq P\left[\xi < b\right] + P\left[(Z_b - Z_a)^2 < \varepsilon\right] < \frac{1}{M}.
\end{aligned}$$

This shows a), and the lemma is proved. □

Lemma 3.4 *Let $(Z_t)_{t \in [a,b]}$ be a continuous stochastic process. If for all $L > 0$ there exist $\mathbb{F}^Z$-stopping times $a = \tau_0 \leq \ldots \leq \tau_n = b$, such that*

$$P\left[\sum_{j=0}^{n-1} \left(Z_{\tau_{j+1}} - Z_{\tau_j}\right)^2 < L\right] < \frac{1}{L}, \tag{3.8}$$

then there exists for all $M > 0$, *a* $\beta \in \mathbf{S}(\mathbb{F}^Z)$ *such that*

a) $P\left[(\beta \cdot Z)_b < M\right] < \frac{1}{M}$,
b) $\inf_{t \in [a,b]} (\beta \cdot Z)_b \geq -\frac{1}{M}$ *and*
c) $|\beta| \leq \frac{1}{M}$.

Proof Let $M > 0$. Since Z is continuous,

$$\xi_N := \inf\{t \in [a,b] : |Z_t - Z_a| \geq N\} \quad (\text{we set } \inf \emptyset = b) \tag{3.9}$$

is for all $N > 0$ an $\mathbb{F}^Z$-stopping time and $\{\xi_N < b\} \to \emptyset$, as $N \to \infty$. Therefore there exists an $N \geq 2$, such that

$$P\left[\xi_N < b\right] < \frac{1}{2M}\,. \tag{3.10}$$

By assumption (3.8) there exist $\mathbb{F}^Z$-stopping times $a = \tau_0 \leq \ldots \leq \tau_n = b$, such that

$$P\left[\sum_{j=0}^{n-1} \left(Z_{\tau_{j+1}} - Z_{\tau_j}\right)^2 < N^2(M^2+1)\right] < \frac{1}{2M}\,. \tag{3.11}$$

It is easy to see that

$$\beta := -\frac{2}{MN^2} \sum_{j=0}^{n-1} \left(Z_{\tau_j} - Z_a\right) 1_{(\tau_j, \tau_{j+1}]} 1_{[0,\xi_N]}$$

is in $\mathbf{S}(\mathbb{F}^Z)$ and satisfies c). For all $t \in [a,b]$,

$$(\beta \cdot Z)_t = \frac{1}{MN^2}\left[\sum_{j=0}^{n-1} \left(Z_{\tau_{j+1} \wedge t \wedge \xi_N} - Z_{\tau_j \wedge t \wedge \xi_N}\right)^2 - \left(Z_{t \wedge \xi_N} - Z_a\right)^2\right]. \tag{3.12}$$

This together with (3.9) implies b). From (3.12), (3.10) and (3.11) it follows that

$$\begin{aligned}
&P\left[(\beta \cdot Z)_b < M\right] \\
&= P\left[\frac{1}{MN^2}\left\{\sum_{j=0}^{n-1} \left(Z_{\tau_{j+1} \wedge \xi_N} - Z_{\tau_j \wedge \xi_N}\right)^2 - \left(Z_{\xi_N} - Z_a\right)^2\right\} < M\right] \\
&\leq P\left[\sum_{j=0}^{n-1} \left(Z_{\tau_{j+1} \wedge \xi_N} - Z_{\tau_j \wedge \xi_N}\right)^2 < M^2N^2 + N^2\right] \\
&\leq P\left[\xi_N < b\right] + P\left[\sum_{j=0}^{n-1} \left(Z_{\tau_{j+1}} - Z_{\tau_j}\right)^2 < N^2(M^2+1)\right] < \frac{1}{M}\,.
\end{aligned}$$

This shows a), and the lemma is proved. □

Lemma 3.5 *Let B^H be a fBm and $T, p, q > 0$. Then:*

a) $n^{pH-1-q} \sum_{j=0}^{n-1} |B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T}|^p \overset{(n\to\infty)}{\longrightarrow} 0$ *in* L^1

b) $n^{pH-1+q} \sum_{j=0}^{n-1} |B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T}|^p \overset{(n\to\infty)}{\longrightarrow} \infty$ *in probability, i.e. for all* $L > 0$ *there exists an* n_0 *such that for all* $n \geq n_0$ *,* $P\left[n^{pH-1+q} \sum_{j=0}^{n-1} |B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T}|^p < L\right] < \frac{1}{L}$

Proof See Lemma 2.1 in Cheridito (2001b). □

Proof of Theorem 3.1 By self-similarity of B^H it is enough to prove Theorem 3.1 for $T = 1$.

(i) $H \in (\frac{1}{2}, 1)$, $\tilde{Y}_t = \nu(t) + \sigma B^H_t$, $t \in [0,1]$:
It is clear that $(\tilde{Y}_t)_{t\in[0,1]}$ satisfies (3.1). Since the function ν is Lipschitz and almost all paths of $(B^H_t)_{t\in[0,1]}$ are Hölder continuous of order α for every $\alpha \in (\frac{1}{2}, H)$, it follows that

$$\max_{t\in[0,1]} \sum_{j=0}^{n-1} \left(\tilde{Y}_{\frac{j+1}{n}\wedge t} - \tilde{Y}_{\frac{j}{n}\wedge t}\right)^2 \overset{(n\to\infty)}{\longrightarrow} 0 \text{ almost surely.} \tag{3.13}$$

This shows that $(\tilde{Y}_t)_{t\in[0,1]}$ also satisfies (3.2). Thus, it follows from Lemma 3.3 that for all $n \in \mathbb{N}$, there exists a $\beta(n) \in \mathbf{S}(\mathbb{F}^{\tilde{Y}})$ such that

$$\text{a) } P\left[\left(\beta(n)\cdot\tilde{Y}\right)_1 < c\right] < \tfrac{1}{n} \quad \text{and}$$
$$\text{b) } \inf_{t\in[0,1]} \left(\beta(n)\cdot\tilde{Y}\right)_t \geq -\tfrac{1}{n}\,.$$

For every $n \in \mathbb{N}$,

$$\xi_n := \inf\left\{t \,:\, \left(\beta(n)\cdot\tilde{Y}\right)_t = c\right\} \quad (\text{we set } \inf\emptyset = 1)$$

is an $\mathbb{F}^Z$-stopping time, and for $\vartheta^1(n) := \beta(n)1_{[0,\xi_n]} \in \mathbf{S}(\mathbb{F}^Z)$ we have

$$\text{a) } P\left[\left(\vartheta^1(n)\cdot\tilde{Y}\right)_1 = c\right] > 1 - \tfrac{1}{n} \quad \text{and}$$
$$\text{b) } \inf_{t\in[0,1]} \left(\vartheta^1(n)\cdot\tilde{Y}\right)_t \geq -\tfrac{1}{n}\,.$$

(ii) $H \in (\frac{1}{2}, 1)$, $\tilde{Y}_t = \exp\left(\nu(t) + \sigma B^H_t\right), t \in [0,1]$:
$(\tilde{Y}_t)_{t\in[0,1]}$ satisfies (3.1), and it is clear that (3.13) still holds true. Hence, $(\tilde{Y}_t)_{t\in[0,1]}$ also satisfies (3.2), and the proof can be concluded as in case (i).

(iii) $H \in (0, \frac{1}{2})$, $\tilde{Y}_t = \nu(t) + \sigma B^H_t$, $t \in [0,1]$:
To show that $(\tilde{Y}_t)_{t\in[0,1]}$ satisfies (3.8) we choose an $L > 0$. It follows from Lemma 3.5 a) that

$$\frac{1}{n}\sum_{j=0}^{n-1} \left|B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}\right| \overset{(n\to\infty)}{\longrightarrow} 0 \text{ in } L^1.$$

Hence,

$$\sum_{j=0}^{n-1} 2 \left| (\nu(\tfrac{j+1}{n}) - \nu(\tfrac{j}{n})) \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}} \right) \right|$$

$$\leq 2 \left\| \nu' \right\|_\infty \frac{1}{n} \sigma \sum_{j=0}^{n-1} \left| B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}} \right| \stackrel{(n \to \infty)}{\longrightarrow} 0 \text{ in } L^1.$$

In particular, there exists an $n_1 \in \mathbb{N}$, such that for all $n \geq n_1$,

$$P \left[\sum_{j=0}^{n-1} \left| 2(\nu(\tfrac{j+1}{n}) - \nu(\tfrac{j}{n})) \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}} \right) \right| > L \right] < \frac{1}{2L} \, .$$

On the other hand, Lemma 3.5 b) implies that there exists an $n_2 \in \mathbb{N}$, such that for all $n \geq n_2$,

$$P \left[\sum_{j=0}^{n-1} \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}} \right)^2 < 2L \right] < \frac{1}{2L} \, .$$

Hence, for all $n \geq \max(n_1, n_2)$,

$$\begin{aligned}
& P \left[\sum_{j=0}^{n-1} \left(\tilde{Y}_{\frac{j+1}{n}} - \tilde{Y}_{\frac{j}{n}} \right)^2 < L \right] \\
& \leq P \left[\sum_{j=0}^{n-1} \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}} \right)^2 + 2(\nu(\tfrac{j+1}{n}) - \nu(\tfrac{j}{n})) \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}} \right) < L \right] \\
& \leq P \left[\sum_{j=0}^{n-1} \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}} \right)^2 < 2L \right] \\
& \quad + P \left[\sum_{j=0}^{n-1} 2 \left| (\nu(\tfrac{j+1}{n}) - \nu(\tfrac{j}{n})) \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}} \right) \right| > L \right] < \frac{1}{L} \, .
\end{aligned}$$

This shows that $(\tilde{Y}_t)_{t \in [0,1]}$ satisfies (3.8). By Lemma 3.4 there exists for all $n \in \mathbb{N}$, a $\beta(n) \in \mathbf{S}(\mathbb{F}^Z)$ such that

a) $P \left[\left(\beta(n) \cdot \tilde{Y} \right)_1 < c \right] < \frac{1}{n}$
b) $\inf_{t \in [0,1]} \left(\beta(n) \cdot \tilde{Y} \right)_t \geq -\frac{1}{n}$
c) $|\beta(n)| \leq \frac{1}{n}$.

Having shown this, we can construct $\vartheta^1(n)$ as in (i). By c) we get $\left| \vartheta^1(n) \right| \leq \frac{1}{n}$.

(iv) $H \in (0, \frac{1}{2})$, $\quad \tilde{Y}_t = \exp\left(\nu(t) + \sigma B_t^H\right), t \in [0,1]$:
Since $(\tilde{Y}_t)_{t\in[0,1]}$ is positive and continuous, $\min_{t\in[0,1]} \tilde{Y}_t > 0$. Therefore, there exists an $\varepsilon > 0$ such that

$$P\left[\min_{t\in[0,1]} \tilde{Y}_t \le \varepsilon\right] < \frac{1}{2L}\,.$$

It follows from what we have shown in the proof of (iii) that there exists an $n \in \mathbb{N}$, such that

$$P\left[\sum_{j=0}^{n-1}\left(\ln \tilde{Y}_{\frac{j+1}{n}} - \ln \tilde{Y}_{\frac{j}{n}}\right)^2 < \frac{1}{\varepsilon^2}L\right] < \frac{1}{2L}\,.$$

Since for all j,

$$\left|\tilde{Y}_{\frac{j+1}{n}} - \tilde{Y}_{\frac{j}{n}}\right| \ge \left(\min_{t\in[0,1]} \tilde{Y}_t\right)\left|\ln \tilde{Y}_{\frac{j+1}{n}} - \ln \tilde{Y}_{\frac{j}{n}}\right|,$$

we obtain

$$\begin{aligned} &P\left[\sum_{j=0}^{n-1}\left(\tilde{Y}_{\frac{j+1}{n}} - \tilde{Y}_{\frac{j}{n}}\right)^2 < L\right] \\ &\le P\left[\min_{t\in[0,1]} \tilde{Y}_t \le \varepsilon\right] + P\left[\sum_{j=0}^{n-1}\left(\ln \tilde{Y}_{\frac{j+1}{n}} - \ln \tilde{Y}_{\frac{j}{n}}\right)^2 < \frac{1}{\varepsilon^2}L\right] < \frac{1}{L}. \end{aligned}$$

This shows that $(\tilde{Y}_t)_{t\in[0,1]}$ satisfies (3.8). Thus, $\vartheta^1(n)$ can be constructed as in (iii). Again, $\left|\vartheta^1(n)\right| \le \frac{1}{n}$. This completes the proof of the theorem. □

Proof of Theorem 3.2 Since B^H is self-similar, it is enough to prove the theorem for $T = 1$. We split $(0,1]$ into the subintervals

$$I_n := (a_n = 1 - 2^{1-n}, b_n = 1 - 2^{-n}]\,,\ n \in \mathbb{N}\,.$$

By $\tilde{Y}^n$ we denote the restriction of $\tilde{Y}$ to I_n and by $\mathbb{F}^{\tilde{Y}^n} = (\mathcal{F}_t^{\tilde{Y}^n})_{t\in I_n}$ the filtration generated by $\tilde{Y}^n$. Note that $\mathcal{F}_t^{\tilde{Y}^n} \subset \mathcal{F}_t^{\tilde{Y}}$ for all $n \in \mathbb{N}$ and $t \in I_n$.

Since B^H has stationary increments, it follows from Theorem 3.1 that there exists for all $n \in \mathbb{N}$, a $\beta(n) \in \mathbf{S}(\mathbb{F}^{\tilde{Y}^n})$ such that

a) $P\left[\left(\beta(n)\cdot\tilde{Y}^n\right)_{b_n} < c + \frac{1}{c}\right] < \frac{1}{n}$

b) $\inf_{t\in I_n}\left(\beta(n)\cdot\tilde{Y}^n\right)_t \ge -\frac{2^{-n}}{c}\,.$

For

$$\beta := \sum_{n=1}^{\infty} \beta(n) 1_{I_n}\,,$$

$$\xi := \inf \left\{ t \in [0,1] \, : \, \left(\beta \cdot \tilde{Y} \right)_t = c \right\} \quad \text{(we set } \inf \emptyset = 1)$$

is an $\mathbb{F}^{\tilde{Y}}$-stopping time. It follows from a) and b) that $P[\xi < 1] = 1$. Therefore, $\vartheta^1 := \beta 1_{[0,\xi]}$ belongs to $\mathbf{aS}(\mathbb{F}^{\tilde{Y}})$ and $(\vartheta^0, \vartheta^1)$ with

$$\vartheta^0_t := \left(\vartheta^1 \cdot \tilde{Y} \right)_t - \vartheta^1_t \tilde{Y}_t \, , \, t \in [0,T] \, ,$$

is a $\frac{1}{c}$-admissible c-arbitrage in $\Theta^{\mathbf{aS}}_{\text{sf,adm}}(\mathbb{F}^{\tilde{Y}})$. In the cases (iii) and (iv) all $\beta(n)$'s can be chosen such that $|\beta(n)| \leq \frac{1}{c}$. Then, $\left|\vartheta^1\right| \leq \frac{1}{c}$ too, and the theorem is proved. □

Remarks 3.6
1) More generally, conclusions a) and b) of Theorem 3.1 hold whenever the discounted stock price $(\tilde{Y}_t)_{t \in [0,T]}$ satisfies conditions (3.1) and (3.2) of Lemma 3.3. If $(\tilde{Y}_t)_{t \in [0,T]}$ fulfils condition (3.8) of Lemma 3.4, then a), b) and c) of Theorem 3.1 are valid. In particular, condition (3.2) is fulfilled by all processes with vanishing quadratic variation and condition (3.8) by all processes with infinite quadratic variation. For different generalizations of Lemma 3.5 see e.g. Shao (1996), Takashima (1989) or Kôno and Maejima (1991). Shao (1996) contains results on p-variation of Gaussian processes with stationary increments. Takashima (1989) gives sample path properties of ergodic self-similar processes, and in Kôno and Maejima (1991), results on Hölder continuity of sample paths of some self-similar stable processes can be found.
2) In a model $(X_t, Y_t)_{t \in [0,T]}$ with strong arbitrage it is possible to super-replicate a European call option with time-T pay-off $C_T = (Y_T - K)^+$, $K > 0$, without initial endowment in the following way: At time 0 one borrows money from the money market account to buy one stock share. Then one applies a strong arbitrage strategy to generate the amount of money needed to pay back the debt without selling the stock share. At time T one owns a stock share and has no debts. This hedges the option. The following example shows that C_T can have a positive super-replication price if the model $(X_t, Y_t)_{t \in [0,T]}$ only admits arbitrage:

Let $(B_t)_{t \in [0,1]}$ be a Brownian motion and $(B^H_t)_{t \in [0,1]}$ a fBm with Hurst parameter $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Let ξ be a random variable that is independent of B and B^H and such that $P[\xi = 0] = P[\xi = 1] = \frac{1}{2}$. Let r, ν and $\sigma > 0$, be constants. Then, the model

$$X_t = \exp(rt) \, , \quad Y_t = \exp\left\{ (r + \nu)t + \sigma \left((1 - \xi) B_t + \xi B^H_t \right) \right\} \, , t \in [0,1] \, ,$$

has arbitrage but no strong arbitrage in $\Theta^{\mathbf{aS}}_{\text{sf,adm}}(\mathbb{F}^Y)$. Is is clear that the super-replication of C_1 with a strategy from $\Theta^{\mathbf{aS}}_{\text{sf,adm}}(\mathbb{F}^Y)$ costs at least the Black-Scholes price.

4 Exclusion of arbitrage

The arbitrage strategies that we constructed in Section 3 act on ever smaller time intervals. They can be excluded by introducing a minimal amount of time $h > 0$ that must lie between two consecutive transactions.

Definition 4.1 *Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a filtration and $h > 0$. We define*

$$\mathbf{S}^h(\mathbb{F}) := \left\{ g_0 1_{\{0\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j,\tau_{j+1}]} \in \mathbf{S}(\mathbb{F}) \, : \, \forall j \, , \, \tau_{j+1} \geq \tau_j + h \right\} \quad \textit{and}$$

$$\Theta^h_{\mathrm{sf}}(\mathbb{F}) := \left\{ \vartheta = (\vartheta^0, \vartheta^1) \in \Theta^{\mathbf{S}}_{\mathrm{sf}} \, : \, \vartheta^0, \vartheta^1 \in \mathbf{S}^h(\mathbb{F}) \right\} .$$

In the following we will show that none of the models (i)-(iv) of Theorem 3.1 has an arbitrage in $\bigcup_{h>0} \Theta^h_{\mathrm{sf}}(\mathbb{F}^{\tilde{Y}})$.

Lemma 4.2 *Let $(B_t)_{t\geq 0}$ be a Brownian motion and $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Let $(Z_t)_{t\geq 0}$ be a continuous version of the process $\left(\int_0^t (t-s)^{H-\frac{1}{2}} dB_s\right)_{t\geq 0}$. Then, for all $c \geq 0$ and all h and T such that $0 < h \leq T$,*

$$P\left[\inf_{t\in[h,T]} Z_t \geq c\right] = P\left[\sup_{t\in[h,T]} Z_t \leq -c\right] > 0 \, .$$

Proof Let $c \geq 0$ and $0 < h \leq T$. Clearly, $(-Z_t)_{t\geq 0}$ has the same distribution as $(Z_t)_{t\geq 0}$. Hence,

$$P\left[\inf_{t\in[h,T]} Z_t \geq c\right] = P\left[\sup_{t\in[h,T]} Z_t \leq -c\right] .$$

We denote by μ_W the Wiener measure on $(C[0,T], \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra generated by the cylinder sets. It follows from Lévy's modulus of continuity for Brownian motion (see e.g. Theorem I.2.7 in Revuz and Yor (1999)) that $\mu_W[\hat{\Omega}] = 1$, where $\hat{\Omega} :=$

$$\left\{ \omega \in C[0,T] \, : \, \omega(0) = 0 \ \text{ and } \ \forall t \in [0,T] \, , \, \lim_{s\to t} \frac{\omega(t) - \omega(s)}{\sqrt{|t-s| \log\left(\frac{1}{|t-s|}\right)}} = 0 \right\} .$$

For every $\omega \in \hat{\Omega}$, $\int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s)$ can for all $t \geq 0$, be defined as a Riemann-Stieltjes integral which is continuous in t (this can be proved like Proposition 1.3 in Cheridito (2001a)). Hence, we have

$$P\left[\inf_{t\in[h,T]} Z_t \geq c\right] = \mu_W\left[\omega \in \hat{\Omega} \, : \, \inf_{t\in[h,T]} \int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s) \geq c\right] .$$

Let us first assume $H \in (\frac{1}{2}, 1)$. In this case we set

$$m := \frac{H + \frac{1}{2}}{h^{H+\frac{1}{2}}} \left(c + T^{H-\frac{1}{2}}\right) \ \text{ and } \ A_m := \left\{ \omega \in \hat{\Omega} \, : \, \sup_{t\in[0,T]} |\omega_m(t)| \leq 1 \right\} ,$$

where ω_m is given by $\omega_m(t) := \omega(t) - mt$, $t \in [0,T]$. By Girsanov's Theorem there exists a probability measure μ_m on $\hat{\Omega}$ that is equivalent to μ_W such that $(\omega_m(t))_{t\in[0,T]}$ is a Brownian motion under μ_m. It is well known that $\mu_m[A_m] > 0$. Equivalence of μ_W and μ_m implies that also

$$\mu_W[A_m] > 0\,. \tag{4.1}$$

For all $\omega \in \hat{\Omega}$ and $t \in [0,T]$,

$$\begin{aligned}\int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s) &= \int_0^t \omega(s)(H-\frac{1}{2})(t-s)^{H-\frac{3}{2}} ds \\ &= (H-\frac{1}{2})\int_0^t \omega_m(s)(t-s)^{H-\frac{3}{2}} ds + (H-\frac{1}{2})\int_0^t ms(t-s)^{H-\frac{3}{2}} ds \\ &= (H-\frac{1}{2})\int_0^t \omega_m(s)(t-s)^{H-\frac{3}{2}} ds + m\frac{t^{H+\frac{1}{2}}}{H+\frac{1}{2}}\end{aligned}$$

For $\omega \in A_m$, we obtain for all $t \in [h,T]$ the following estimates:

$$(H-\frac{1}{2})\int_0^t \omega_m(s)(t-s)^{H-\frac{3}{2}} ds \geq -(H-\frac{1}{2})\int_0^t (t-s)^{H-\frac{3}{2}} ds \geq -T^{H-\frac{1}{2}}\,,$$

and, by our choice of m,

$$m\frac{t^{H+\frac{1}{2}}}{H+\frac{1}{2}} = \left(\frac{t}{h}\right)^{H+\frac{1}{2}} \left(c + T^{H-\frac{1}{2}}\right) \geq c + T^{H-\frac{1}{2}}\,.$$

Hence,

$$\int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s) ds \geq -T^{H-\frac{1}{2}} + c + T^{H-\frac{1}{2}} = c\,.$$

It follows that

$$A_m \subset \left\{\inf_{t\in[h,T]} \int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s) \geq c\right\}\,,$$

which together with (4.1) proves the lemma for $H \in (\frac{1}{2}, 1)$.

For $H \in (0, \frac{1}{2})$, the proof is slightly more delicate. Since all $\omega \in \hat{\Omega}$ are Hölder continuous of order α for every $\alpha < \frac{1}{2}$, there exists a constant $\delta > 0$ such that

$$\mu_W\left[A(\frac{1}{2}, \delta)\right] > 0\,,$$

where

$$A(\frac{1}{2}, \delta) := \left\{\omega \in \hat{\Omega} : \sup_{t\in[0,T]} |\omega(t)| \leq \frac{1}{2} \text{ and } \sup_{t,s\in[0,T]} \frac{|\omega(t)-\omega(s)|}{(t-s)^{\frac{1}{2}-\frac{H}{2}}} \leq \delta\right\}$$

We set

$$m := \frac{H+\frac{1}{2}}{h^{H+\frac{1}{2}}}\left[c + (\frac{1}{2} - H)\frac{2\delta}{H} T^{\frac{H}{2}} + \frac{1}{2} h^{H-\frac{1}{2}}\right]\,,$$

$$\omega_m(t) := \omega(t) - mt\,,\; t \in [0,T]$$

and define μ_m as before. Furthermore, we set

$$A_m(\frac{1}{2},\delta) := \left\{\omega \in \hat{\Omega} \,:\, \omega_m \in A(\frac{1}{2},\delta)\right\}.$$

Since $(\omega_m(t))_{t\in[0,T]}$ is a Brownian motion under μ_m,

$$\mu_m\left[A_m(\frac{1}{2},\delta)\right] = \mu_W\left[A(\frac{1}{2},\delta)\right] > 0\,,$$

and therefore,

$$\mu_W\left[A_m(\frac{1}{2},\delta)\right] > 0\,. \tag{4.2}$$

For $\omega \in \hat{\Omega}$ and $t \in [h,T]$, we can write

$$\begin{aligned}
&\int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s) = \int_0^t (t-s)^{H-\frac{1}{2}} d\left[\omega(s) - \omega(t)\right] \\
&= (\frac{1}{2} - H)\int_0^t \left[\omega(t) - \omega(s)\right](t-s)^{H-\frac{3}{2}} ds + t^{H-\frac{1}{2}}\omega(t) \\
&= (\frac{1}{2} - H)\int_0^t \left[\omega_m(t) - \omega_m(s)\right](t-s)^{H-\frac{3}{2}} ds + t^{H-\frac{1}{2}}\omega_m(t) + m\frac{t^{H+\frac{1}{2}}}{H+\frac{1}{2}}\,.
\end{aligned}$$

If $\omega \in A_m(\frac{1}{2},\delta)$ and $t \in [h,T]$, we can estimate the three preceding terms as follows:

$$\begin{aligned}
&(\frac{1}{2} - H)\int_0^t \left[\omega_m(t) - \omega_m(s)\right](t-s)^{H-\frac{3}{2}} ds \\
&\geq -(\frac{1}{2} - H)\int_0^t \delta(t-s)^{\frac{H}{2}-1} ds \geq -(\frac{1}{2} - H)\frac{2\delta}{H}T^{\frac{H}{2}}\,, \\
&t^{H-\frac{1}{2}}\omega_m(t) \geq -\frac{1}{2}h^{H-\frac{1}{2}}\,, \\
&m\frac{t^{H+\frac{1}{2}}}{H+\frac{1}{2}} = \left(\frac{t}{h}\right)^{H+\frac{1}{2}}\left[c + (\frac{1}{2} - H)\frac{2\delta}{H}T^{\frac{H}{2}} + \frac{1}{2}h^{H-\frac{1}{2}}\right] \\
&\geq c + (\frac{1}{2} - H)\frac{2\delta}{H}T^{\frac{H}{2}} + \frac{1}{2}h^{H-\frac{1}{2}}\,.
\end{aligned}$$

It follows that

$$\int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s) \geq c\,.$$

This and (4.2) prove the lemma for $H \in (0, \frac{1}{2})$. □

Theorem 4.3 *Let B^H be a fBm with $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Let $T \in (0, \infty)$, $\sigma > 0$ and $\nu : [0,T] \to \mathbb{R}$ be a measurable function such that $\sup_{t \in [0,T]} |\nu(t)| < \infty$. Consider the two cases*

$$\text{(i)}\ \tilde{Y}_t = \nu(t) + \sigma B^H_t,\ t \in [0,T]$$
$$\text{(ii)}\ \tilde{Y}_t = \exp\left(\nu(t) + \sigma B^H_t\right),\ t \in [0,T]$$

If

$$\vartheta^1 = g_0 1_{\{0\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j, \tau_{j+1}]} \in \bigcup_{h>0} \mathbf{S}^h(\mathbb{F}^{\tilde{Y}})$$

and there exists a $j \in \{1, \dots, n-1\}$ with $P[g_j \neq 0] > 0$, then in case (i)*,*

$$P\left[\left(\vartheta^1 \cdot \tilde{Y}\right)_T \leq -c\right] > 0 \quad \textit{for all } c \geq 0\,,$$

and in case (ii)*,*

$$P\left[\left(\vartheta^1 \cdot \tilde{Y}\right)_T < 0\right] > 0\,.$$

Proof For notational simplicity we give the proof for $\tilde{Y}_t = B^H_t$ and $\tilde{Y}_t = \exp(B^H_t)$. The generalizations to the cases (i) and (ii) are obvious. To prove the theorem for $\tilde{Y}_t = B^H_t$ we fix an $h > 0$, and consider a

$$\vartheta^1 = g_0 1_{\{0\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j, \tau_{j+1}]} \in \mathbf{S}^h(\mathbb{F}^{B^H})\,,$$

such that there exists a $j \in \{1, \dots, n-1\}$ with $P[g_j \neq 0] > 0$. If

$$k = \max\left\{j \in \{1, \dots, n-1\} \,:\, P[g_j \neq 0] > 0\right\}\,,$$

then

$$\left(\vartheta^1 \cdot B^H\right)_T = \sum_{j=1}^{k} g_j \left(B^H_{\tau_{j+1}} - B^H_{\tau_j}\right) \text{ almost surely.}$$

Let $c \geq 0$. It is clear that

$$\begin{aligned} & P\left[\sum_{j=1}^{k} g_j \left(B^H_{\tau_{j+1}} - B^H_{\tau_j}\right) \leq -c\right] \\ \geq\ & P\left[\sum_{j=1}^{k-1} g_j \left(B^H_{\tau_{j+1}} - B^H_{\tau_j}\right) + \sup_{t \in [h,T]} g_k \left(B^H_{\tau_k + t} - B^H_{\tau_k}\right) \leq -c\right]. \end{aligned} \tag{4.3}$$

Let

$$\hat{\Omega} := \left\{\omega \in C(\mathbb{R}) \,:\, \omega(0) = 0 \text{ and } \forall t \geq \mathbb{R}\,, \lim_{s \to t} \frac{\omega(t) - \omega(s)}{\sqrt{|t-s|} \log\left(\frac{1}{|t-s|}\right)} = 0\right\}\,,$$

$\mathcal{B}$ the σ-algebra of subsets of $\hat{\Omega}$ that is generated by the cylinder sets and P the Wiener measure on $(\hat{\Omega}, \mathcal{B})$. Without loss of generality we can assume that $(B^H_t)_{t \geq 0}$ is defined on $(\hat{\Omega}, \mathcal{B}, P)$ by the improper Riemann-Stieltjes integrals

$$B^H_t(\omega) = \int_{-\infty}^{t} \left[(t-s)^{H-\frac{1}{2}} - 1_{\{s \leq 0\}} (-s)^{H-\frac{1}{2}} \right] d\omega(s) \, , \, t \geq 0 \tag{4.4}$$

(see Proposition 1.3 in Cheridito (2001a)). We define the filtration $\mathbb{F}^{\hat{\Omega}} = (\mathcal{F}^{\hat{\Omega}}_t)_{t \in [0,T]}$ by

$$\mathcal{F}^{\hat{\Omega}}_t := \sigma \left\{ \left\{ \omega \in \hat{\Omega} \, : \, \omega(s) \leq a \right\} \, : \, -\infty < s \leq t \, , \, a \in \mathbb{R} \right\} .$$

It is clear that $\mathbb{F}^{\hat{\Omega}}$ contains the filtration $\mathbb{F}^{B^H} = (\mathcal{F}^{B^H}_t)_{t \in [0,T]}$, which is given by

$$\mathcal{F}^{B^H}_t := \sigma \left\{ B^H_s \, : \, 0 \leq s \leq t \right\} .$$

Therefore the $\mathbb{F}^{B^H}$-stopping times $\tau_1, \ldots \tau_k$, are also $\mathbb{F}^{\hat{\Omega}}$-stopping times. In the following we split each function $\omega \in \hat{\Omega}$ at the time point $\tau_k(\omega)$. We set

$$\pi_1 \omega(s) := \omega(s) 1_{(-\infty, \tau_k(\omega)]}(s) \, , \, s \in \mathbb{R} \, ,$$

$$\pi_2 \omega(s) := \omega(\tau_k(\omega) + s) - \omega(\tau_k(\omega)) \, , \, s \geq 0 \, ,$$

and let

$$\Omega_1 := \left\{ \pi_1(\omega) \in \mathbb{R}^{\mathbb{R}} \, : \, \omega \in \hat{\Omega} \right\} ,$$

$\mathcal{B}_1$ the σ-algebra of subsets of Ω_1 that is generated by the cylinder sets,

$$\Omega_2 := \left\{ \pi_2(\omega) \in C[0, \infty) \, : \, \omega \in \hat{\Omega} \right\}$$

and $\mathcal{B}_2$ the σ-algebra of subsets of Ω_2 that is generated by the cylinder sets. It can easily be checked that the mapping $\pi_1 : (\hat{\Omega}, \mathcal{B}) \rightarrow (\Omega_1, \mathcal{B}_1)$ is $\mathcal{F}^{\hat{\Omega}}_{\tau_k}$-measurable. On the other hand, it follows from Theorem I.32 of Protter (1990) that $(\pi_2 \omega(s))_{s \geq 0}$ is a Brownian motion under P which is independent of $\mathcal{F}^{\hat{\Omega}}_{\tau_k}$. It can be seen from (4.4) that for all $\omega \in \hat{\Omega}$ and $t \in [h, T]$,

$$\left(\sum_{j=1}^{k-1} g_j \left(B^H_{\tau_{j+1}} - B^H_{\tau_j} \right) + g_k \left(B^H_{\tau_k + t} - B^H_{\tau_k} \right) \right) (\omega) = U_t(\pi_1 \omega, \pi_2 \omega)$$

where for $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ and $t \in [h, T]$,

$$U_t(\omega_1, \omega_2) := U^0(\omega_1) + g_k(\omega_1) \left(U^1_t(\omega_1) + U^2_t(\omega_2) \right) ,$$

$$U^0(\omega_1) := \left(\sum_{j=1}^{k-1} g_j \left(B^H_{\tau_{j+1}} - B^H_{\tau_j} \right) \right) (\omega_1) \, ,$$

$$U^1_t(\omega_1) := \int_{-\infty}^{\tau_k(\omega_1)} \left[(\tau_k(\omega_1) + t - s)^{H-\frac{1}{2}} - (\tau_k(\omega_1) - s)^{H-\frac{1}{2}} \right] d\omega_1(s) \, ,$$

$$U^2_t(\omega_2) := \int_0^t (t-s)^{H-\frac{1}{2}} d\omega_2(s) \, .$$

Since $(U_t)_{t\in[h,T]}$ is a continuous stochastic process on $(\Omega_1 \times \Omega_2, \mathcal{B}_1 \otimes \mathcal{B}_1)$, the set

$$A := \left\{ (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 \,:\, \sup_{t\in[h,T]} U_t(\omega_1, \omega_2) \le -c \right\}$$

is $\mathcal{B}_1 \otimes \mathcal{B}_2$-measurable. It follows from Proposition A.2.5 of Lamberton and Lapeyre (1996) that for almost every $\omega \in \hat{\Omega}$,

$$\mathrm{E}\left[1_A(\pi_1, \pi_2) \mid \mathcal{F}^{\hat{\Omega}}_{\tau_k} \right](\omega) = \phi(\pi_1 \omega) \,,$$

where the mapping $\phi : \Omega_1 \to \mathbb{R}$ is given by $\phi(\omega_1) := \mathrm{E}\left[1_A(\omega_1, \pi_2)\right], \omega_1 \in \Omega_1$. Since $U^1_t(\omega_1)$ is for all $\omega \in \Omega_1$ continuous in t, $\sup_{t\in[h,T]} U^1_t(\omega_1)$ is for all $\omega_1 \in \Omega_1$ finite. Therefore and since $(\pi_2\omega(t))_{t\ge 0}$ is a Brownian motion under P, it follows from Lemma 4.2 that for fixed $\omega_1 \in \Omega_1$,

$$\begin{aligned}
\phi(\omega_1) &= P\left[\sup_{t\in[h,T]} U_t(\omega_1, \pi_2) \le -c \right] \\
&\ge P\left[U^0(\omega_1) + \sup_{t\in[h,T]} g_k(\omega_1) U^1_t(\omega_1) + \sup_{t\in[h,T]} g_k(\omega_1) U^2_t(\pi_2) \le -c \right] > 0 \,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P\left[\sum_{j=1}^{k-1} g_j \left(B^H_{\tau_{j+1}} - B^H_{\tau_j} \right) + \sup_{t\in[h,T]} g_k \left(B^H_{\tau_k + t} - B^H_{\tau_k} \right) \le -c \right] \\
&= \mathrm{E}\left[1_A(\pi_1, \pi_2)\right] = \mathrm{E}\left[\mathrm{E}\left[1_A(\pi_1, \pi_2) \mid \mathcal{F}^{\hat{\Omega}}_{\tau_k} \right] \right] = \mathrm{E}\left[\phi \circ \pi_1\right] > 0 \,.
\end{aligned}$$

This and (4.3) prove the theorem in the case $\tilde{Y}_t = B^H_t$.
If $\tilde{Y}_t = \exp(B^H_t)$, let us assume there exists an $h > 0$ and a

$$\vartheta^1 = g_0 1_{\{0\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j, \tau_{j+1}]} \in \mathbf{S}^h(\mathbb{F}^{B^H})$$

such that there exists a $j \in \{1, \ldots, n-1\}$ with $P\left[g_j \neq 0\right] > 0$ and $(\vartheta^1 \cdot \tilde{Y})_T \ge 0$ almost surely. If

$$k = \min \left\{ l \,:\, P\left[g_l \neq 0\right] > 0 \text{ and } \sum_{j=1}^{l} g_j \left(\mathrm{e}^{B^H_{\tau_{j+1}}} - \mathrm{e}^{B^H_{\tau_j}} \right) \ge 0 \text{ a.s.} \right\} ,$$

then either

$$\sum_{j=1}^{k-1} g_j \left(\mathrm{e}^{B^H_{\tau_{j+1}}} - \mathrm{e}^{B^H_{\tau_j}} \right) = 0 \text{ almost surely, or}$$

$$P\left[\sum_{j=1}^{k-1} g_j\left(\mathrm{e}^{B^H_{\tau_{j+1}}} - \mathrm{e}^{B^H_{\tau_j}}\right) < 0\right] > 0\,.$$

In both cases, $P[C] > 0$ for

$$C := \left\{\sum_{j=1}^{k-1} g_j\left(\mathrm{e}^{B^H_{\tau_{j+1}}} - \mathrm{e}^{B^H_{\tau_j}}\right) \leq 0\right\}\,.$$

With the same method that we used in the first part of the proof one can deduce from Lemma 4.2 that for almost all $\omega \in C$,

$$P\left[\sum_{j=1}^{k-1} g_j\left(\mathrm{e}^{B^H_{\tau_{j+1}}} - \mathrm{e}^{B^H_{\tau_j}}\right) + \sup_{t\in[h,T]} g_k\left(\mathrm{e}^{B^H_{\tau_k+t}} - \mathrm{e}^{B^H_{\tau_k}}\right) < 0 \,\middle|\, \mathcal{F}^{\hat{\Omega}}_{\tau_k}\right](\omega) > 0\,,$$

which implies that

$$\begin{aligned} &P\left[\sum_{j=1}^{k} g_j\left(\mathrm{e}^{B^H_{\tau_{j+1}}} - \mathrm{e}^{B^H_{\tau_j}}\right) < 0\right] \\ &\geq P\left[\sum_{j=1}^{k-1} g_j\left(\mathrm{e}^{B^H_{\tau_{j+1}}} - \mathrm{e}^{B^H_{\tau_j}}\right) + \sup_{t\in[h,T]} g_k\left(\mathrm{e}^{B^H_{\tau_k+t}} - \mathrm{e}^{B^H_{\tau_k}}\right) < 0\right] > 0\,. \end{aligned}$$

This contradicts our assumption and therefore, completes the second part of the proof. □

It follows from Theorem 4.3 that if trading strategies are restricted to the class $\bigcup_{h>0}\Theta^h_{\mathrm{sf}}(\mathbb{F}^{\tilde{Y}})$, then in case (i) there exist no non-trivial admissible strategies and in particular no FLVR, and in case (ii) there exists no arbitrage. An inspection of the proof of Theorem 4.3 shows that in case (ii), a $\vartheta \in \bigcup_{h>0}\Theta^h_{\mathrm{sf}}(\mathbb{F}^{\tilde{Y}})$ can only be admissible if ϑ^1 is almost surely non-negative.

It follows from similar arguments to the ones in the proof of Theorem 4.3 that in both cases (i) and (ii) the cheapest way to super-replicate a European call option with a strategy $\vartheta \in \bigcup_{h>0}\Theta^h_{\mathrm{sf}}(\mathbb{F}^{\tilde{Y}})$ is to buy the stock. In particular, in both cases (i) and (ii) of Theorem 4.3 the model $(X_t, Y_t)_{t\in[0,T]}$ is incomplete when trading strategies are restricted to $\bigcup_{h>0}\Theta^h_{\mathrm{sf}}(\mathbb{F}^{\tilde{Y}})$.

References

1. Cheridito, P.: Regularizing fractional Brownian motion with a view towards stock price modelling. Diss. ETH No. 14051, www.math.ethz.ch/∼dito (2001a)
2. Cheridito, P.: Mixed fractional Brownian motion. Bernoulli **7**(6), 913-934 (2001b)
3. Cutland, N.J., Kopp P.E., Willinger W.: Stock price returns and the Joseph effect: A fractional version of the Black-Scholes model. Progress in Probability **36**, 327-351 (1995)

4. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math. Ann. **300**, 463-520 (1994)
5. Harrison, J.M., Pliska, S.P.: Martingales and stochastic integrals in the theory of continuous trading. Stoch. Proc. Appl. **11**, 215-260. (1981)
6. Kôno, N., Maejima, M.: Hölder continuity of sample paths of some self-similar stable processes. Tokyo Journal of Mathematics **14**(1), 93-100 (1991)
7. Lamberton, D., Lapeyre, B.: Stochastic calculus applied to finance. Chapman & Hall (1996)
8. Liptser, R.Sh., Shiryaev A.N.: Theory of martingales. Kluwer Acad. Publ., Dordrecht (1989)
9. Protter, P.: Stochastic integration and differential equations. Berlin: Springer-Verlag (1990)
10. Revuz, D., Yor, M.: Continuous martingales and Brownian motion. Springer (1999)
11. Rogers, L.C.G.: Arbitrage with fractional Brownian motion. Mathematical Finance **7**, 95-105 (1997)
12. Salopek, D.M.: Tolerance to arbitrage. Stochastic Processes and their Applications **76**, 217-230 (1998)
13. Samorodnitsky, G., Taqqu, M.S.: Stable non-Gaussian random processes. Stochastic models with infinite variance. Chapman & Hall, New York (1994)
14. Shao, Q.M.: p-variation of Gaussian processes with stationary increments. Studia Sci. Math. Hungar. **31**, 237-247 (1996)
15. Shiryaev, A.N.: On arbitrage and replication for fractal models. Research Report **20**, MaPhySto, Department of Mathematical Sciences, University of Aarhus, Denmark (1998)
16. Takashima, K.: Sample path properties of ergodic self-similar processes. Osaka J. Math. **26**, 159-189 (1989)
17. Willinger, W., Taqqu, M.S., Teverovsky, V.: Stock market prices and long-range dependence. Finance Stochast. **3**(1), 1-13 (1999)
18. Young, L.C.: An inequality of the Hölder type, connected with Stieltjes integration. Acta Math. (Sweden) **67**, 251-282 (1936)