Affine Markov chain models of multifirm credit migration [1]

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Abstract. This paper introduces and explores a natural extension of the Chen–Filipovic affine models for credit migration, credit spreads and credit default correlation. The essential addition proposed here is to introduce a Markov chain for the "credit rating" of each firm, which are independent conditioned on a stochastic time change. The stochastic time change is then combined with other stochastic factors, here the interest rate and the recovery rate, into a multidimensional affine process. The resulting general framework has the computational effectiveness of the Chen–Filipovic models, but without certain of their conceptual drawbacks. This paper, as the first of the series, aims to illustrate the potential of the general framework by exploring a minimal implementation which is still capable of combining stochastic interest rates, stochastic recovery rates and the multifirm default process. Already within this minimal version we see very good reproduction of essential features such as credit spread curves, default correlations and multifirm default distributions.

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1 Introduction

The purpose of this paper is to present a simple version of a modeling framework capable of describing the joint credit worthiness of many firms, and consequently to enable computation of general credit sensitive contingent claims. The modeling framework has several attractive characteristics. First, it is flexible enough to include many features of real default processes. Second, all essential computations can be reduced to closed formulas involving characteristic functions of affine processes which are either explicitly known or can be efficiently computed numerically via standard ODE solvers. Finally, the modeling assumptions are stated clearly and can be scrutinized independently of the consequent computations.

The credit models of Chen and Filipovic [2, 3] showed how a wide range of characteristic features of multifirm default processes, including realistic credit spreads and correlation effects such as contagion [9] and infectious defaults [4], can be brought into an affine modeling framework. Affine models have been very intensely studied [6, 5] and it is well understood how even high dimensional problems can be very efficiently computed. Despite their conceptual and computational elegance, these models have some drawbacks which limit their applicability. One shortcoming of the Chen–Filipovic family of models is that when the contagion effect is included, a firm will continue to influence other firms even after its default. As more and more firms default over time, this effect grows, and the model blows up at large times. It also seems that even over short time periods this effect will dominate computations involving a large number of firms, for example the pricing of collateralized debt obligations (CDOs). Another variation of their approach found in [3], includes a more detailed modeling of the credit migration of a firm. This approach is appealing and effective for a single firm, but it leads to difficulties when extended to the multifirm case, again because firms continue after their default to influence the remaining firms.

Another natural modeling approach to credit migration as developed by [13, 1, 6, 14] and others is to regard the credit rating class of each firm as a Markov chain with stochastic transition intensities. Default is represented in such models as an extra rating class which is "absorbing". Many variations have been studied which differ in the modeling of the rating intensities.

The basic model we propose here combines the essential features of the affine approach with a particular version of the Markov chain credit migration picture. The computational efficiency of the model is based on a central mathematical observation: a continuous time Markov chain subjected to an independent stochastic time change which is in the class of integrated positive affine processes results in a model where all primary quantities are expressible in terms of the affine generator. This trick performs for us a role similar to the "basic trick" at the heart of [2].

Based on this mathematical insight, we can move on to create a range of models of increasing complexity which capture in a plausible way more and more features of the real credit world.

It is worth explaining in more detail the credit migration picture underlying such models. In the simplest possible versions, there is a single time change for the entire credit market. Viewed in stochastic time, all firms undergo their credit migration independently with identical transition intensities, eventually jumping to the absorbing default state. Viewed in real time however, the stochastic clock variously speeds up or slows down each firm's migration process, thereby raising or lowering their true default intensity. The result is a complex structure of credit spreads and positive default correlations. A more general framework allows for a multiplicity of stochastic time changes, possibly describing different sectors of the economy.

The literature for credit risk modeling is huge and rapidly growing. For a review of the broad classification of credit risk models see [8, 7, 14]: one sees that our model is in the class of reduced form (intensity based) models.

The organization of this paper is as follows. The main ingredients of the framework are the processes which govern credit migration, interest rate, time change and recovery, and are described in detail in section 2. Section 3 defines the basic computational building blocks which underlie all future computations. Section 4 focusses on the computation of pure rating transition and default probabilities. Section 5 gives general formulas for defaultable zero coupon bonds and credit default swaps. Section 6 derives formulas for the default correlation in a basic two firm context. To give an impression of the character of these model, we describe a two firm "toy model" in section 7 and present graphs of a wide range of important quantities. This toy model is based on three independent affine factors: a Cox–Ingersoll–Ross (CIR) diffusion, a pure jump intensity, and a third which generates stochastic time jumps. These factors jointly drive the interest rate, time change and recovery. The appendix shows the detailed computations underlying the basic building blocks.

In related work, [11] demonstrates that this modeling framework has an extremely tractable limit when N, the number of firms, becomes arbitrarily large. This paper then goes on to compute large N approximations of CDO pricing.

Notation: Vectors in this article will be written in bold font as $\mathbf{x} = \{x^1, x^2, ..., x^n\}$ with subscripts on top.

Unless indicated otherwise, probabilities should be understood as risk–neutral probabilities.

2 The model setup

The basic model of a single credit risky firm is built from the following main ingredients:

(a) The spot interest rate r_t and the recovery rate R_t;

(b) A stochastic time change τ_t, a nondecreasing process in time;

(c) A process $\tilde{Y}_t$, a continuous time Markov chain on $\{1, 2, \ldots, K\}$, with 1 an absorbing state.

In our models, $\tilde{Y}$ is always independent of r, R, τ. The Markov states of $\tilde{Y}$ can be thought of as "credit ratings". For example, if $K = 8$ we can identify them with Moody's or Standard and Poor's ratings classes:
$\{1, 2, \ldots, 8\} \leftrightarrow \{$'default', CCC, B, BB, BBB, A, AA, AAA$\}$.

These ingredients combine to make the real time credit variables:

(d) The real time credit migration process Y_t, defined as $\tilde{Y}_t$ subordinated by τ_t

$$Y_t = \tilde{Y}_{\tau_t}. \tag{2.1}$$

(e) The default time t^*, defined as the first time the process Y_t hits the absorbing state 1.

The joint laws of the time change, interest rate and recovery processes are derived from a set of independent underlying factors, each of which is a positive process. For simplicity we take as factors a two–dimensional affine process $\mathbf{Z}_t = (Z_t^1, Z_t^2)$ and a jump process $\tau_t^{(j)}$. Let $\mathcal{F}_t^Z$ be the σ-algebra generated by $\{\mathbf{Z}_s | s \le t\}$ and $\mathcal{F}_t^\tau$ be the σ-algebra generated by $\{\mathbf{Z}_s, \tau_s^{(j)} | s \le t\}$. Then we define

- The interest rate $r_t = \langle \mathbf{M}_r \cdot \mathbf{Z}_t \rangle$;
- The time change as a sum of absolutely continuous and jump components

$$\tau_t = \tau_t^{(ac)} + \tau_t^{(j)} = \int_0^t \langle \mathbf{M}_\tau \cdot \mathbf{Z}_s \rangle ds + m_\tau \tau_t^{(j)}; \tag{2.2}$$

- The recovery process

$$R_t = e^{-\langle \mathbf{M}_R \cdot \mathbf{Z}_t \rangle}. \tag{2.3}$$

Here $\mathbf{M}_r$, $\mathbf{M}_\tau$ and $\mathbf{M}_R$ are vectors from $\mathbb{R}_+^2$ and $m_\tau \ge 0$.

Our specification of $Z^1, Z^2, \tau^{(j)}$ illustrates the range of possibilities: Z^1 is a CIR process with Markov generator

$$\mathcal{L}_{Z^1} f(x) = a(1-x)f'(x) + cxf''(x), \tag{2.4}$$

and Z^2 is an affine process with jumps defined by its Markov generator

$$\mathcal{L}_{Z^2} f(x) = \lambda_2 (f(x+h) - f(x)) - h\lambda_2 x f'(x). \tag{2.5}$$

Finally, $\tau_t^{(j)}$ (a jump part of the time change) is a Poisson process with intensity λ_3 and jump size λ_3^{-1}:

$$\tau_t^{(j)} = \lambda_3^{-1} \Pi(\lambda_3 t). \tag{2.6}$$

Note that Z^2 undergoes jumps of size $h > 0$ with intensity λ_2 and then decays exponentially (with the speed of decay given by $h\lambda_2$). Also $Z_t^1, Z_t^2, \tau_t^{(j)}$ are normalized to have long term means of $1, 1, t$ respectively.

The law of $\tilde{Y}$ is defined by its transition semigroup $\mathcal{P}_Y(t)$, the $K \times K$ matrix valued function such that:

$$P_{0,y}(\tilde{Y}_t = j) = (\mathcal{P}_Y)_{yj}(t), \quad y, j \in \{1, \dots, K\} \tag{2.7}$$

We suppose that $\mathcal{P}_Y(t) = e^{t\mathcal{L}_Y}$ where the infinitesimal generator has the form:

$$\mathcal{L}_Y = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ L_{21} & -L_{22} & L_{23} & \dots & L_{2,K} \\ L_{31} & L_{32} & -L_{33} & \dots & L_{3,K} \\ \dots & \dots & \dots & \dots & \dots \\ L_{K,1} & L_{K,2} & L_{K,3} & \dots & -L_{K,K} \end{pmatrix} \tag{2.8}$$

with $L_{ij} \ge 0$ and $L_{ii} = \sum_{j \ne i} L_{ij}$.

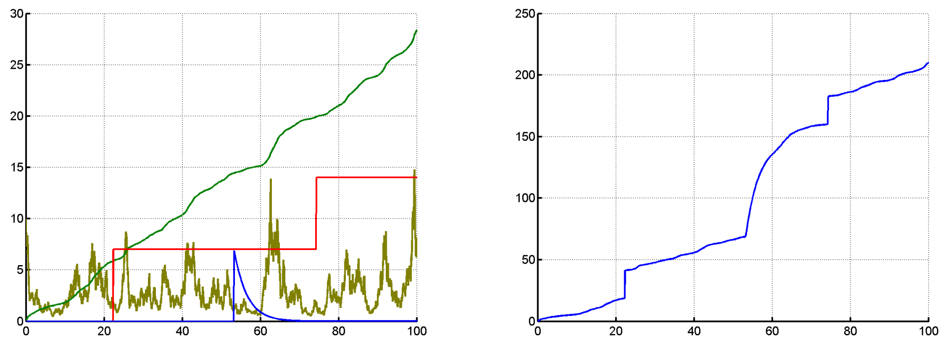

Figure 1: simulation of the time change factors

3 Building blocks

The essential pricing formulas we derive will all involve the following basic building blocks.

Functions G_1 and G_2 are defined by

$$G_1(t,\mathbf{z};\mathbf{u},\mathbf{v}) = E_{0,\mathbf{z}}\left[e^{-\int_0^t \langle \mathbf{u}\cdot\mathbf{Z}_s\rangle ds} e^{-\langle \mathbf{v}\cdot\mathbf{Z}_t\rangle}\right] \tag{3.1}$$

and

$$G_2(t,\mathbf{z};\mathbf{u},\mathbf{v},\mathbf{w}) = E_{0,\mathbf{z}}\left[e^{-\int_0^t \langle \mathbf{u}\cdot\mathbf{Z}_s\rangle ds}\langle \mathbf{w}\cdot\mathbf{Z}_t\rangle e^{-\langle \mathbf{v}\cdot\mathbf{Z}_t\rangle}\right] = -\langle \mathbf{w},\nabla_{\mathbf{v}}\mathbf{G_1}\rangle. \tag{3.2}$$

Here $\mathbf{z}$, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^2$.

When Z_1, Z_2 are general affine processes, G_1 is exponential–affine in $\mathbf{Z}$:

$$G_1(t,\mathbf{z};\mathbf{u},\mathbf{v}) = e^{\Phi(T-t)+\langle\Psi(T-t)\cdot\mathbf{Z}_t\rangle}, \tag{3.3}$$

where the functions $\Psi(t) = \{\Psi^1(t),\Psi^2(t)\}$ and $\Phi(t)$ are efficiently computable by solving generalized Riccati differential equations. With our simple specification of Z_1, Z_2, these functions are given by closed formulas which can be found in Appendix A.

The Laplace transform of $\tau_t^{(j)}$ is given explicitly by

$$G_\tau(t;v) = E\left[e^{-v\tau_t^{(j)}}\right] = \exp\left(\lambda_3 t(e^{-v/\lambda_3}-1)\right). \tag{3.4}$$

Lastly, turning to the Markov semigroup, we suppose that the transition matrix and its generator can be diagonalized:

$$\mathcal{P}_Y(t) = Qe^{tD}Q^{-1}, \quad \mathcal{L}_Y = QDQ^{-1}, \tag{3.5}$$

where $D = \mathrm{diag}\{\alpha_1,\alpha_2,\ldots,\alpha_K\}$ is a diagonal matrix and $Q = (q_{ij})_{i,j=1\ldots K}$ is a matrix whose columns are the corresponding eigenvectors of $\mathcal{L}_Y$. Let the elements of Q^{-1} be denoted as $Q^{-1} = (\tilde{q}_{ij})_{i,j=1\ldots K}$. Therefore

$$(\mathcal{P}_Y)_{yj}(t) = \sum_{i=1}^K q_{yi}\tilde{q}_{ij}e^{\alpha_i t} \tag{3.6}$$

4 Rating transition probabilities

The transition probabilities of the real–time rating migration process Y_t, in particular the distribution of the time of default, are now computable in terms of the basic building blocks. This illustrates how the Markov chain combined with the affine stochastic time change forms a tight yet versatile structure.

Lemma 1. *The rating transition probabilities for the process Y_t are given by*

$$P_{0,\mathbf{z},y}(Y_t = j) = \sum_{i=1}^{K} q_{yi}\tilde{q}_{ij} G_1(t, \mathbf{z}; -\alpha_i \mathbf{M}_\tau, \mathbf{0}) G_\tau(t, -\alpha_i m_\tau). \tag{4.1}$$

Proof.

$$\begin{aligned} P_{0,\mathbf{z},y}(Y_t = j) &= E_{0,\mathbf{z},y}[I\{\tilde{Y}_{\tau_t} = j\}] = E_{0,\mathbf{z}}[E_{0,\mathbf{z},y}[I\{\tilde{Y}_{\tau_t} = j\}|\mathcal{F}^\tau_\infty]] \\ &= E_{0,\mathbf{z}}[(\mathcal{P}_Y)_{yj}(\tau_t)] = E_{0,\mathbf{z}}\left[\sum_{i=1}^{K} q_{yi}\tilde{q}_{ij} e^{\alpha_i \tau_t}\right] = \sum_{i=1}^{K} q_{yi}\tilde{q}_{ij} E_{0,\mathbf{z}}\left[e^{\alpha_i \tau_t}\right]. \end{aligned}$$

The result follows since

$$E_{0,\mathbf{z}}\left[e^{\alpha_i \tau_t}\right] = E_{0,\mathbf{z}}\left[e^{\alpha_i \int_0^t \langle \mathbf{M}_\tau \cdot \mathbf{Z}_s\rangle ds + \alpha_i m_\tau \tau_t^{(j)}}\right] = G_1(t, \mathbf{z}; -\alpha_i \mathbf{M}_\tau, \mathbf{0}) G_\tau(t, -\alpha_i m_\tau).$$

□

Default time is the first time Y_t hits the absorbing state 1 which means $\{t^* < t\} \equiv \{Y_t = 1\}$, and so the survival probability and density of the default time are computed from (4.1) with $j = 1$:

Corollary 2. *The survival probability at time $t > 0$ is given by*

$$P_{0,\mathbf{z},y}(t^* < t) = \sum_{i=1}^{K} q_{yi}\tilde{q}_{i1} G_1(t, \mathbf{z}; -\alpha_i \mathbf{M}_\tau, \mathbf{0}) G_\tau(t, -\alpha_i m_\tau). \tag{4.2}$$

The probability density function of default is given by

$$\begin{aligned} &\frac{d}{dt} P_{0,\mathbf{z},y}(t^* < t) \\ &= \sum_{i=1}^{K} q_{yi}\tilde{q}_{i1} \left[\alpha_i G_2(t, \mathbf{z}; -\alpha_i \mathbf{M}_\tau, \mathbf{0}, \mathbf{M}_\tau) + G_1(t, \mathbf{z}; -\alpha_i \mathbf{M}_\tau, \mathbf{0}) \lambda_3 (e^{\alpha_i m_\tau/\lambda_3} - 1)\right] G_\tau(t, -\alpha_i m_\tau). \end{aligned}$$

Proof. The density function is obtained by differentiating (4.2) in t and noting that

$$\begin{aligned} &\frac{d}{dt} G_1(t, \mathbf{z}; -\alpha_i \mathbf{M}_\tau, \mathbf{0}) = \frac{d}{dt} E_{0,\mathbf{z}}\left[e^{\alpha_i \int_0^t \langle \mathbf{M}_\tau \cdot \mathbf{Z}_s\rangle ds}\right] \\ &= E_{0,\mathbf{z}}\left[\alpha_i \langle \mathbf{M}_\tau \cdot \mathbf{Z}_t\rangle e^{\alpha_i \int_0^t \langle \mathbf{M}_\tau \cdot \mathbf{Z}_s\rangle ds}\right] = \alpha_i G_2(t, \mathbf{z}; -\alpha_i \mathbf{M}_\tau, \mathbf{0}, \mathbf{M}_\tau). \end{aligned}$$

□

5 Pricing defaultable securities

In this section we derive formulas for the basic default risky securities: zero–coupon bonds and credit default swaps. We first recall that $B_t(T)$, the price at time t of a *riskless* zero–coupon bond with maturity T, is given by

$$B_t(T) = E_t\left[e^{-\int_t^T r_s ds}\right] = G_1(T-t, \mathbf{Z}_t; \mathbf{M}_r, \mathbf{0}).$$

The price at time t of a *defaultable* zero–coupon bond with maturity T with a specified recovery process X will be denoted $B_t^{(d)}(T, X)$. In particular $B_t^{(d)}(T, 0)$ is the price of a defaultable bond with zero recovery.

Lemma 3. *The price of a defaultable bond with zero recovery is given by*

$$\begin{aligned} B_t^{(d)}(T,0) &= E_{t,\mathbf{z},y}\left[e^{-\int_t^T r_s ds} I\{t^* > T\}\right] \\ &= B_t(T) - \sum_{i=1}^K q_{yi}\tilde{q}_{i1} G_1(T-t, \mathbf{z}; \mathbf{M}_r - \alpha_i \mathbf{M}_\tau, \mathbf{0}) G_\tau(T-t, -\alpha_i m_\tau). \end{aligned}$$

Proof. We find

$$\begin{aligned} E_{t,\mathbf{z},y}\left[e^{-\int_t^T r_s ds} I\{t^* > T\}\right] &= E_{t,\mathbf{z},y}\left[e^{-\int_t^T r_s ds}(1 - I\{Y_T = 1\})\right] \\ &= B_t(T) - E_{t,\mathbf{z},y}\left[e^{-\int_t^T r_s ds} I\{Y_T = 1\}\right] \end{aligned}$$

where the second expectation can be computed as

$$\begin{aligned} E_{t,\mathbf{z},y}\left[e^{-\int_t^T r_s ds} I\{Y_T = 1\}\right] &= E_{t,\mathbf{z}}\left[e^{-\int_t^T r_s ds} E_{t,y}\left[I\{Y_T = 1\} | \mathcal{F}_\infty^\tau\right]\right] \\ &= E_{t,\mathbf{z}}\left[e^{-\int_t^T r_s ds} \sum_{i=1}^K q_{yi}\tilde{q}_{i1} e^{\alpha_i(\tau_T - \tau_t)}\right] \\ &= \sum_{i=1}^K q_{yi}\tilde{q}_{i1} E_{t,\mathbf{z}}\left[e^{-\int_t^T \langle \mathbf{M}_r \cdot \mathbf{Z}_s\rangle ds} e^{\alpha_i \int_t^T \langle \mathbf{M}_\tau \cdot \mathbf{Z}_s\rangle ds + \alpha_i(\tau_T^{(\mathrm{j})} - \tau_t^{(\mathrm{j})})}\right] \\ &= \sum_{i=1}^K q_{yi}\tilde{q}_{i1} G_1(T-t, \mathbf{z}; \mathbf{M}_r - \alpha_i \mathbf{M}_\tau, \mathbf{0}) G_\tau(T-t; -\alpha_i m_\tau). \end{aligned}$$

□

In the next lemma we compute the price of a credit default swap (CDS). One party, the insured, pays a constant rate, called the *CDS spread*, up to $t^* \wedge T$, the minimum of default time t^* and maturity. If default happens before maturity T, the other party, the insurer, pays the recovery fraction of the riskless bond with maturity T:

$$X_{t^*} = R_{t^*} B_{t^*}(T). \tag{5.1}$$

This version of recovery is called *recovery of treasury*; other recovery specifications include *recovery of par* and *recovery of market value*.

The CDS spread is given by $CDS = \frac{W^S}{V^R}$ where the price of the *premium leg* which pays at rate 1 while $t < t^* \wedge T$ is

$$V^R = E_{t,\mathbf{z},y}\left[\int_t^T e^{-\int_t^s r_u du} I\{t^* > s\} ds\right] \tag{5.2}$$

and the price of the *insurance leg* is

$$W^S = E_{t,\mathbf{z},y}\left[e^{-\int_t^{t^*} r_s ds} X_{t^*} I\{t^* < T\}\right]. \tag{5.3}$$

Lemma 4. *The price of the premium leg is given by*

$$V^R = \int_t^T B_t^{(d)}(s,0) ds, \tag{5.4}$$

The price of the insurance leg is

$$\begin{aligned} W^S &= = \sum_{i=1}^K q_{yi}\tilde{q}_{i1} \int_t^T G_\tau(u-t; -\alpha_i m_\tau) e^{\Phi(T-u)} \\ &\times \ [\alpha_i G_2(u-t, \mathbf{z}; \mathbf{M}_r - \alpha_i \mathbf{M}_\tau, \mathbf{M}_R - \Psi(T-u), \mathbf{M}_\tau) + \\ &\qquad + G_1(u-t, \mathbf{z}; \mathbf{M}_r - \alpha_i \mathbf{M}_\tau, \mathbf{M}_R - \Psi(T-u)) \lambda_3 (e^{\alpha_i m_\tau/\lambda_3} - 1)] du. \end{aligned}$$

Here $\Psi(t) = \{\Psi^1(t), \Psi^2(t)\}$ *and* $\Phi(t)$ *are the functions given by (A.3) in the Appendix which price the default–free bond:*

$$B_t(T) = e^{\Phi(T-t) + \langle \Psi(T-t) \cdot Z_t \rangle}.$$

Proof. The premium leg is straightforward. The insurance leg is more complicated:

$$\begin{aligned} & E_{t,\mathbf{z},y}\left[e^{-\int_t^{t^*} r_s ds} X_{t^*} I\{t^* < T\}\right] = E_{t,\mathbf{z}}\left[E_{t,y}\left[e^{-\int_t^{t^*} r_s ds} X_{t^*} I\{t^* < T\} | \mathcal{F}_\infty^Z\right]\right] \\ &= E_{t,\mathbf{z}}\left[\int_t^T E_{t,y}\left[e^{-\int_t^u r_s ds} X_u | \mathcal{F}_\infty^Z\right] \frac{d}{du} P_{t,y}(t^* < u | \mathcal{F}_\infty^Z) du\right] \\ &= E_{t,\mathbf{z}}\left[\int_t^T e^{-\int_t^u r_s ds} X_u \frac{d}{du}\left(\sum_{i=1}^K q_{yi}\tilde{q}_{i1} e^{\alpha_i \int_t^u \langle \mathbf{M}_\tau \cdot \mathbf{Z}_s \rangle ds} G_\tau(u-t; -\alpha_i m_\tau)\right) du\right] \\ &= \sum_{i=1}^K q_{yi}\tilde{q}_{i1} \int_t^T G_\tau(u-t; -\alpha_i m_\tau) \times \\ &\qquad \times E_{t,\mathbf{z}}\left[e^{-\int_t^u r_s ds} X_u e^{\alpha_i \int_t^u \langle \mathbf{M}_\tau \cdot \mathbf{Z}_s \rangle ds} \left(\alpha_i \langle \mathbf{M}_\tau \cdot \mathbf{Z}_u \rangle + \lambda_3 (e^{\alpha_i m_\tau/\lambda_3} - 1)\right)\right] du \end{aligned}$$

To complete the derivation, we note that the interest rate is $r_u = \langle \mathbf{M}_r \cdot \mathbf{Z}_u \rangle$ and the recovery process X_u can be computed as

$$X_u = R_u B_u(T) = e^{-\langle \mathbf{M}_R \cdot \mathbf{Z}_u \rangle} e^{\Phi(T-u) + \langle \Psi(T-u) \cdot Z_u \rangle} = e^{\Phi(T-u)} e^{-\langle (\mathbf{M}_R - \Psi(T-u)) \cdot \mathbf{Z}_u \rangle}.$$

Finally, the remaining expectation is done using the definitions of G_1 and G_2. □

When the recovery process is RT, a defaultable bond is equivalent to a zero–recovery defaultable bond plus the insurance leg of a CDS.

Corollary 5. *The price at time t of defaultable bond with stochastic recovery $X_t = R_t B_t(T)$ of maturity T is given by the following expression:*

$$\begin{aligned} B_t^{(d)}(T,X) &= E_{t,\mathbf{z},y}\left[e^{-\int_t^T r_s ds} I\{t^* > T\}\right] + E_{t,\mathbf{z},y}\left[e^{-\int_t^{t^*} r_s ds} X_{t^*} I\{t^* < T\}\right] \\ &= B_t^{(d)}(T,0) + W^S. \end{aligned}$$

The *credit yield spread* on such bonds is defined to be the function

$$h(t,T,X) = \frac{\log B_t^{(d)}(T,X) - \log B_t^{(T)}}{T-t} \tag{5.5}$$

6 A two firm model

The simplest way to extend our credit framework to several firms is to assume:

Assumption 1. Firms are distinguishable in their creditworthiness at time t only by their ratings class at that time.

While this assumption certainly fails for real ratings systems such as Moody's and Standard and Poor's (since their ratings do not attempt to measure instantaneous credit worthiness), it should be true for an ideal rating system.

The consequence of this assumption is that different firms are described by Markov chains with the same transition probabilities, subordinated by the same stochastic time change τ_t. Therefore the ingredients for the two firm model are:

(a) Two independent processes $\tilde{Y}_t^1, \tilde{Y}_t^2$, which are finite Markov chains on $\{1, 2, \ldots, K\}$, with 1 an absorbing state, and with identical generators $\mathcal{L}$.

(b) The spot interest process r_t, the recovery process R_t and the stochastic time change process τ_t, as for the one firm model.

Then the real time credit variables are:

(c) The real time credit migration processes for firms 1 and 2 are

$$Y_t^1 = \tilde{Y}_{\tau_t}^1, \quad Y_t^2 = \tilde{Y}_{\tau_t}^2. \tag{6.1}$$

(d) The default times t_1^*, t_2^* of companies 1 and 2, defined as the first time the corresponding process Y_t^i hits the absorbing state 1.

In this framework, all credit derivatives on a single firm are computed as before. Here we compute the correlation between default times of the two firms and their joint distribution of defaults.

The correlation between events $\{t_i^* < t\}$ is defined to be

$$\rho_{12}(t) = \frac{E_{12}(t) - E_1(t)E_2(t)}{\sqrt{(E_1(t) - E_1(t)^2)(E_2(t) - E_2(t)^2)}}, \tag{6.2}$$

where

$$E_{12}(t) \;=\; E_{0,\mathbf{z},y_1,y_2}\left[I\{t_1^* < t\}I\{t_2^* < t\}\right],$$

and

$$E_n(t) \;=\; E_{0,\mathbf{z},y_n}\left[I\{t_n^* < t\}\right].$$

The following formulas are proved by mimicking the proofs done so far:

Lemma 6. *The above expectations E_1, E_2, E_{12} are given by*

$$\begin{aligned} E_{12}(t) &= E_{0,\mathbf{z},y_1,y_2}\left[I\{Y_t^1 = 1\}I\{Y_t^2 = 1\}\right] \\ &= \sum_{i,j=1}^{K} q_{y_1 i}\tilde{q}_{i1}q_{y_2 j}\tilde{q}_{j1}G_1(t,\mathbf{z};-(\alpha_i+\alpha_j)\mathbf{M}_\tau,\mathbf{0})G_\tau(t;-(\alpha_i+\alpha_j)m_\tau), \end{aligned}$$

and

$$\begin{aligned} E_n(t) &= E_{0,\mathbf{z},y_n}\left[I\{Y_t^n = 1\}\right] \\ &= \sum_{i=1}^{K} q_{y_n i}\tilde{q}_{i1}G_1(t,\mathbf{z};-\alpha_i\mathbf{M}_\tau,\mathbf{0})G_\tau(t;-\alpha_i), \quad n = 1,2. \end{aligned}$$

As a final computation, we find the joint default probability distribution:

Lemma 7. *The joint probability $P_{0,\mathbf{z},y_1,y_2}(t_1^* < s, t_2^* < t)$ is given by*

$$\begin{aligned} P_{0,\mathbf{z},y_1,y_2}(t_1^* < s, t_2^* < t) &= E_{0,\mathbf{z},y_1,y_2}\left[I\{Y_s^1 = 1\}I\{Y_t^2 = 1\}\right] \qquad (6.3) \\ &= \sum_{i,j=1}^{K} q_{y_1 i}\tilde{q}_{i1}q_{y_2 j}\tilde{q}_{j1}E_{0,\mathbf{z}}\left[e^{\alpha_i\tau_s+\alpha_j\tau_t}\right], \end{aligned}$$

where the expectation $E_{0,\mathbf{z},y_1,y_2}\left[e^{\alpha_i\tau_s+\alpha_j\tau_t}\right]$ can be computed explicitly (see Appendix A).

7 A toy model

In this section we will illustrate how the pieces of the model can be specified, and look to see how effectively the resulting simple model performs, both in computational speed and in capturing the range of "stylistic features" of real credit markets.

The first step is to determine the Markov generator $\mathcal{L}_y$, a $K \times K$ stochastic matrix. In view of Assumption 1 of the previous section, we seek transition probabilities of an ideal rating system. Since that does not exist, we instead use the information about this generator derived from published one year transition matrices $\mathcal{P}_1$, based on the observed annualized frequency of rating changes. Here is one published matrix (Standard and Poor Credit Week, 15 April 1996):

$$\mathcal{P}_1 = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.1979 & 0.6485 & 0.1124 & 0.0238 & 0.0130 & 0.0022 & 0.0000 & 0.0022 \\ 0.0520 & 0.0407 & 0.8347 & 0.0648 & 0.0043 & 0.0024 & 0.0011 & 0.0000 \\ 0.0106 & 0.0100 & 0.0884 & 0.8053 & 0.0773 & 0.0067 & 0.0014 & 0.0003 \\ 0.0018 & 0.0012 & 0.0117 & 0.0530 & 0.8693 & 0.0595 & 0.0033 & 0.0002 \\ 0.0006 & 0.0001 & 0.0026 & 0.0074 & 0.0552 & 0.9105 & 0.0227 & 0.0009 \\ 0.0000 & 0.0002 & 0.0014 & 0.0006 & 0.0064 & 0.0784 & 0.9122 & 0.0007 \\ 0.0000 & 0.0000 & 0.0000 & 0.0012 & 0.0006 & 0.0068 & 0.0833 & 0.9081 \end{pmatrix}$$

This matrix, the transition probability for a discrete time Markov chain, is not actually a one year transition matrix for any *continuous time* Markov chain with generator $\mathcal{L}_Y$ (many of the zero entries are inconsistent with this hypothesis). Nevertheless, it is approximately of the correct form: If we compute $\log(P_1)$ by diagonalization we find it to be close to the following Markov generator:

$$\mathcal{L}_Y = \begin{pmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.2400 & -0.4380 & 0.1514 & 0.0262 & 0.0157 & 0.0020 & 0.0000 & 0.0029 \\ 0.0507 & 0.0549 & -0.1889 & 0.0786 & 0.0010 & 0.0024 & 0.0012 & 0.0000 \\ 0.0077 & 0.0107 & 0.1070 & -0.2240 & 0.0924 & 0.0045 & 0.0013 & 0.0003 \\ 0.0012 & 0.0009 & 0.0103 & 0.0629 & -0.1450 & 0.0667 & 0.0028 & 0.0002 \\ 0.0005 & 0.0000 & 0.0022 & 0.0066 & 0.0618 & -0.0969 & 0.0249 & 0.0010 \\ 0.0000 & 0.0002 & 0.0014 & 0.0002 & 0.0045 & 0.0857 & -0.0994 & 0.0007 \\ 0.0000 & 0.0000 & 0.0000 & 0.0014 & 0.0002 & 0.0035 & 0.0918 & -0.0966 \end{pmatrix}$$

Here, it turns out that $\log(P_1)$ has several small but negative off–diagonal elements of the order 10^{-4} which have been put to zero. As discussed in [12], this method doesn't work in general, and several more sophisticated approximate methods are available.

Next we need to choose parameters for the processes $\mathbf{Z}$ and $\tau_t^{(j)}$ as well as coefficients $\mathbf{M}_r, \mathbf{M}_\tau, \mathbf{M}_R$ and m_τ. For our present expository purposes, we fix the parameters entering $\mathbf{Z}$ and $\tau_t^{(j)}$ to be $(a, c, \lambda_2, h, \lambda_3) = (0.2, 0.1, 0.025, 20, 0.05)$ and initial values $Z_1(0) = 1, Z_2(0) = 1, \tau^{(j)}(0) = 0$. We also fix the interest rate and recovery parameters to be $\mathbf{M}_r = (0.02, 0.02),\ \mathbf{M}_R = (0.2, 0.2)$. Finally, to show the range of distinctive behaviors attributable to the stochastic time change, we look at four model variations with differing $\mathbf{M}_\tau, m_\tau$: model A has $\mathbf{M}_\tau = (1, 0), m_\tau = 0$; model B has $\mathbf{M}_\tau = (0.5, 0.5), m_\tau = 0$; model C has $\mathbf{M}_\tau = (0.5, 0), m_\tau = 0.5$. In all four cases, the average speed of the stochastic time change is 2, which leads to risk–neutral default probabilities approximately twice the historical values implied by the matrix $\mathcal{L}_Y$.

Figures 2 to 10 show typical graphs that result for the key quantities computed from the model. Figures 2, 3, 4 show the joint probability of default for two firms both of class BB, for the three models A, B, C. Notice that in model C, the presence of a $\tau^{(j)}$ component implies the measure is singular with respect to Lebesgue measure. Figure 5 plots the term structure of the credit yield spread (5.5) for firms in all rating classes over a 5 year period. This plot is for model A, but models B and C are very similar. Figure 6 plots the term structure of the CDS spread computed using Lemma 4, for firms in all rating classes over a 5 year period. Again, this plot is for model A, but models B and C are very similar. Figures 7 to 10 shows the default correlation of a BB firm against various rating classes as a function of time, for various versions of the three different models A, B, C. We see that the detailed shape of these curves is sensitive to the details of the stochastic time change.

Taken together, these plots illustrate a range of features observed in real market data, and show that the modelling framework has the right qualitative features to be useful for credit risk.

8 Concluding Remarks

This article has introduced a flexible yet computationally efficient model of credit risk. In the simple version presented here, each firm undergoes a credit migration which is correlated with market conditions only through the stochastic time change τ_t. The speed of the single time change process provides a measure of the credit environment experienced by firms at each time: when the speed is high, firms migrate quickly and hence default quickly; and the opposite when the speed is low.

The dynamics built into the model reflects in a plausible way the true dynamics of the market. In focussing on dynamics, our model contrasts with static copula methods for credit risk, which are the current industrial standard models for multifirm credit products. A detailed comparison of the relative advantages of the two frameworks is needed. The simple model we present here produces a wide range of possible behavior, and the graphs we show pass visual inspection to be plausible representation of the real market.

Many generalizations to higher dimensions are possible. By introducing an N–dimensional stochastic time change process, the framework can in principle be calibrated to the characteristics of N individual firms, with a computational complexity of order $O(K^N)$. This can be very feasible when N is not too large. [10] is a preliminary work addressing calibration issues for such models.

For basket credit products written on a large number of names, such as CDOs and basket CDSs, it is natural to distinguish firms only by ratings class. In this case, the computational complexity can be reduced enormously. For example, [11] focusses on the limit of a large number of identically distributed firms. In this limit many basket derivatives can be computed as quickly as basic one firm computations.

In summary, we have introduced a versatile family of credit risk models capable in principle of reproducing most of the important features of real markets. The framework is appears to be deserving of future development.

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Appendix A: Formulas for the building blocks

Since Z^1 and Z^2 are independent, we have

$$G_1(t, \mathbf{z}; \mathbf{u}, \mathbf{v}) = \prod_{i=1}^{2} E_{0,z^i}\left[e^{-u^i \int_0^t Z_s^i ds} e^{-v^i Z_t^i}\right] = \prod_{i=1}^{2} G_1^{(i)}(t, z^i; u^i, v^i) \tag{A.1}$$

The function G_2 can be obtained from G_1 by differentiation

$$\begin{aligned} G_2(t, \mathbf{z}; \mathbf{u}, \mathbf{v}, \mathbf{w}) &= E_{0,\mathbf{z}}\left[e^{-\int_0^t \langle \mathbf{u} \cdot \mathbf{Z}_s \rangle ds} \langle \mathbf{w} \cdot \mathbf{Z}_t \rangle e^{-\langle \mathbf{v} \cdot \mathbf{Z}_t \rangle}\right] \\ &= w^1 G_2^{(1)}(t, z^1; u^1, v^1) G_1^{(2)}(t, z^2; u^2, v^2) + w^2 G_1^{(1)}(t, z^1; u^1, v^1) G_2^{(2)}(t, z^2; u^2, v^2). \end{aligned}$$

where for the process $Z = Z^i$ the function $G_2^{(i)}(t, z; u, v)$ is defined as

$$G_2^{(i)}(t, z; u, v) = E_{0,z}\left[e^{-\int_0^t u Z_s ds} Z_t e^{-v Z_t}\right] = -\frac{\partial}{\partial v} G_1^{(i)}(t, z; u, v).$$

When Z^1 is a CIR process with generator

$$\mathcal{L}_{Z^1} f(x) = (a - bx) f'(x) + cx f''(x),$$

$G_1^{(1)}$ is well known to have the exponential affine form

$$G_1^{(1)}(t, z; u, v) = E_{0,z}\left[e^{-u \int_0^t Z_s ds} e^{-v Z_t}\right] = e^{\phi(t,u,v) + z\psi(t,u,v)}. \tag{A.2}$$

Here the functions ϕ and ψ are explicit:

$$\begin{cases} \psi(t, u, v) = \psi_2 - \left(1 + \frac{c}{\gamma}(v + \psi_1)\left(e^{\gamma t} - 1\right)\right)^{-1} (v + \psi_2), \\ \phi(t, u, v) = a\psi_1 t - \frac{a}{c} \log\left(e^{-\gamma t} + \frac{c}{\gamma}(v + \psi_1)\left(1 - e^{-\gamma t}\right)\right) \end{cases} \tag{A.3}$$

with constants ψ_1, ψ_2 and γ given by

$$\begin{cases} \gamma = \sqrt{b^2 + 4uc} \\ \psi_1 = \frac{b+\gamma}{2c} \\ \psi_2 = \frac{b-\gamma}{2c} \end{cases} \tag{A.4}$$

The function $G_2^{(1)}$ is

$$\begin{aligned} G_2^{(1)}(t, z; u, v) &= -\frac{\partial}{\partial v} G_1^{(1)}(t, z; u, v) = -\frac{\partial}{\partial v} e^{\phi(t,u,v)+z\psi(t,u,v)} \\ &= \Big(1 + \frac{c}{\gamma}(v + \psi_1)(e^{\gamma t} - 1)\Big)^{-1} \Big(\frac{a(e^{\gamma t} - 1)}{\gamma} + ze^{\gamma t}\Big) G_1^{(1)}(t, z; u, v). \end{aligned}$$

When Z^2 has the Markov generator

$$\mathcal{L}_{Z^2} f(x) = \lambda(f(x + h) - f(x)) - bxf'(x)$$

the process is again affine and

$$G_1^{(2)}(t, z; u, v) = E_{0,z}\left[e^{-u\int_0^t Z_s ds} e^{-vZ_t}\right] = e^{\phi(t,u,v)+z\psi(t,u,v)}, \tag{A.5}$$

Now the functions ϕ and ψ are given as solutions to the following system of equations

$$\begin{cases} \frac{d\psi}{dt} = -b\psi - u, \qquad \psi(0, u, v) = -v \\ \\ \frac{d\phi}{dt} = \lambda(e^{h\psi} - 1), \quad \phi(0, u, v) = 0. \end{cases} \tag{A.6}$$

This system can be solved explicitly to give the following expressions

$$\begin{cases} \psi(t, u, v) = \left(\frac{u}{b} - v\right) e^{-bt} - \frac{u}{b} \\ \phi(t, u, v) = \frac{\lambda}{b} e^{-\frac{uh}{b}} (Ei(h(\frac{u}{b} - v)) - Ei(h(\frac{u}{b} - v)e^{-bt})) - \lambda t, \end{cases} \tag{A.7}$$

where $Ei(x)$ is the special function called *exponential integral* and defined as a Cauchy principal value of the integral $\int_{-\infty}^{x} y^{-1} e^y dy$. The function $G_2^{(2)}$ is

$$G_2^{(2)}(t, z; u, v) = \left(\frac{\lambda}{u - bv} e^{-\frac{uh}{b}} \left(e^{h(\frac{u}{b}-v)} - e^{h(\frac{u}{b}-v)e^{-bt}}\right) + ze^{-bt}\right) G_1^{(2)}(t, z; u, v).$$

Finally we derive an explicit formula for the expectation

$$E(s, t) = E_{0,\mathbf{z}}\left[e^{\alpha\tau_s+\beta\tau_t}\right], \tag{A.8}$$

which is the key factor in the formula for joint default probabilities appearing in Lemma 7. If we assume $t > s$ (the argument for $t < s$ is similar) we can rewrite this expectation as

$$\begin{aligned} &E(s, t) = E_{0,\mathbf{z}}\left[e^{\alpha\tau_s+\beta\tau_t}\right] = E_{0,\mathbf{z}}\left[E_s\left[e^{\alpha\tau_s+\beta\tau_t}\right]\right] = \\ &E_{0,\mathbf{z}}\left[e^{\alpha\tau_s+\beta\tau_s} E_s\left[e^{\beta(\tau_t-\tau_s)}\right]\right]. \end{aligned} \tag{A.9}$$

The inner expectation can be computed

$$\begin{aligned} E_s\left[e^{\beta(\tau_t-\tau_s)}\right] &= E_s\left[e^{\beta\int_s^t \langle \mathbf{M}_\tau \cdot \mathbf{Z}_u du\rangle + \beta(\tau_t^{(\mathrm{j})} - \tau_s^{(\mathrm{j})})}\right] \\ &= G_1(t-s, \mathbf{Z}_s, -\beta \mathbf{M}_\tau, \mathbf{0}) G_\tau(t-s; -\beta) = e^{\Phi + \langle \Psi \cdot \mathbf{Z}_s \rangle} G_\tau(t-s; -\beta), \end{aligned} \tag{A.10}$$

where the last equality is true since the process $\mathbf{Z}_t$ is affine. Now we plug formula (A.10) into expression (A.9) for $E(s,t)$ and find that

$$\begin{aligned} E(s,t) &= G_\tau(t-s; -\beta) e^{\Phi} E_{0,\mathbf{z}}\left[e^{(\alpha+\beta)\tau_s} e^{\langle \Psi \cdot \mathbf{Z}_s \rangle}\right] \\ &= G_\tau(t-s; -\beta) e^{\Phi} G_\tau(s; -(\alpha+\beta)) G_1(s, \mathbf{z}; -(\alpha+\beta)\mathbf{M}_\tau, -\Psi). \end{aligned}$$

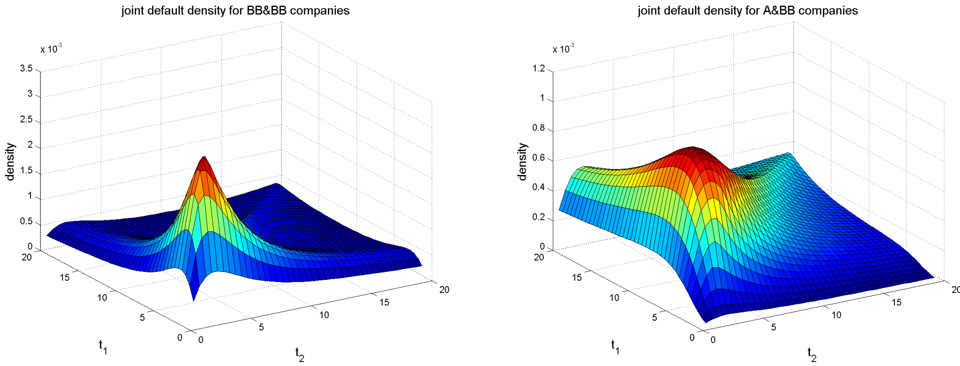

Figure 2: joint default density (a)

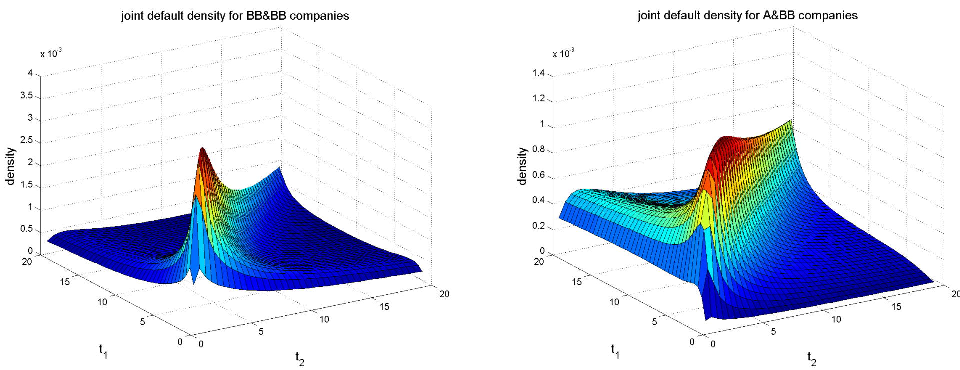

Figure 3: joint default density (b)

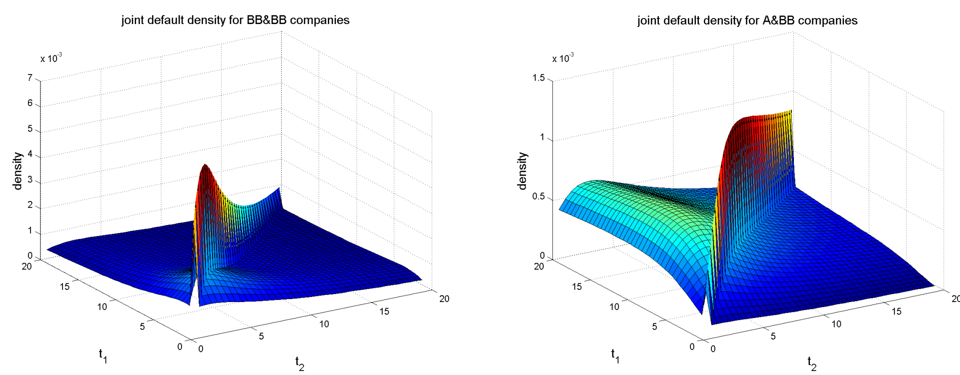

Figure 4: joint default density (c)

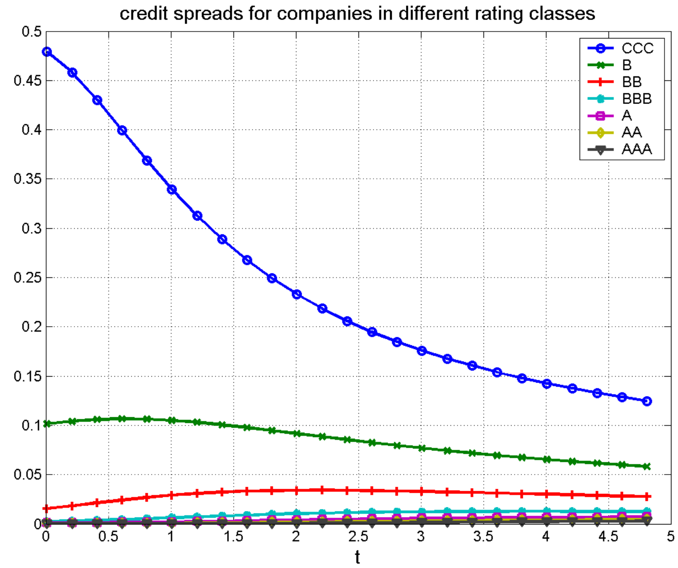

Figure 5: hazard rates

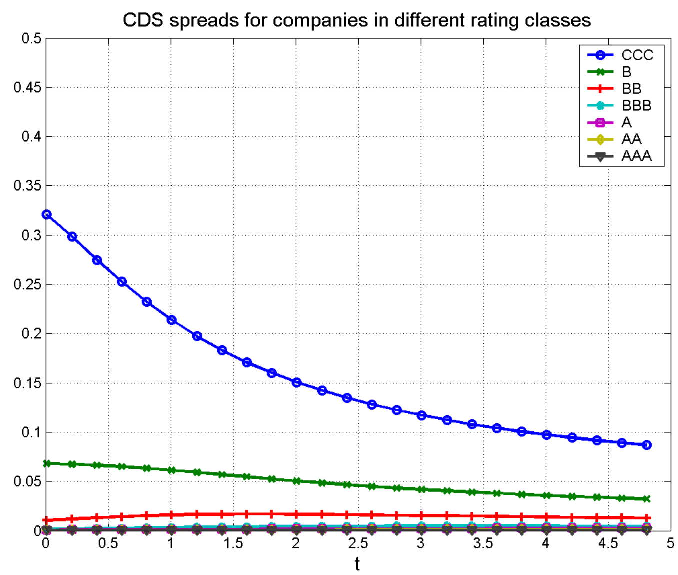

Figure 6: Credit Default Swap

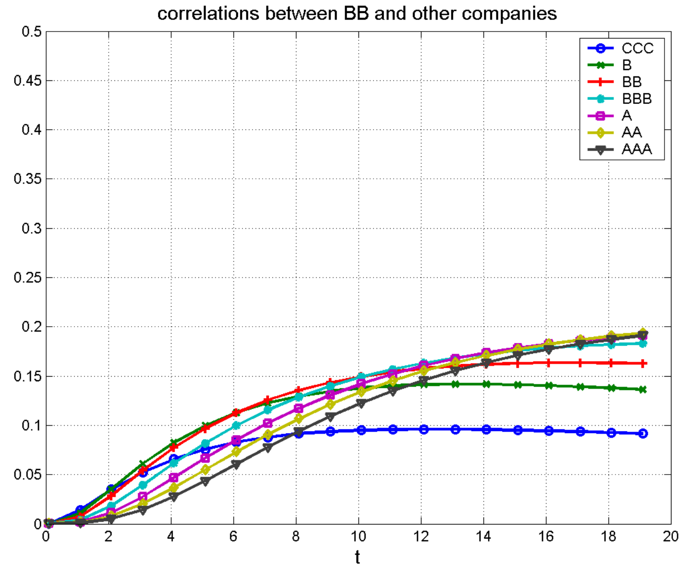

Figure 7: Correlation BB+X (a) CIR diffusion=0.1,0.5,1

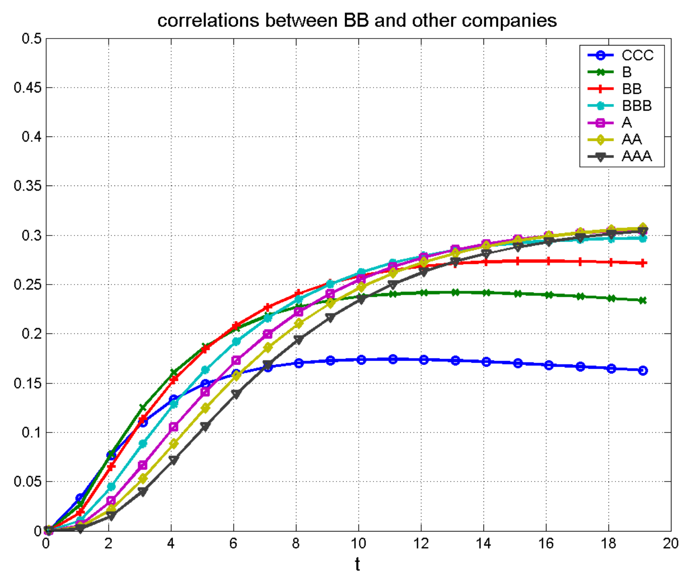

Figure 8: Correlation BB+X (a) CIR diffusion=0.5

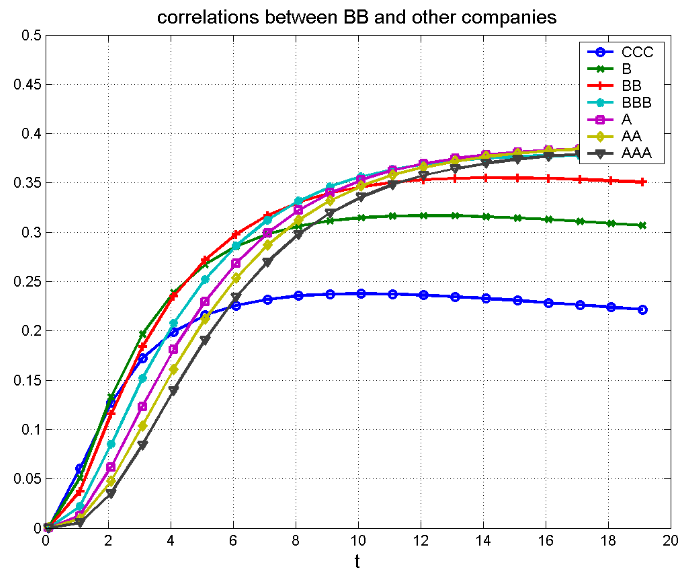

Figure 9: Correlation BB+X (a) CIR diffusion=1

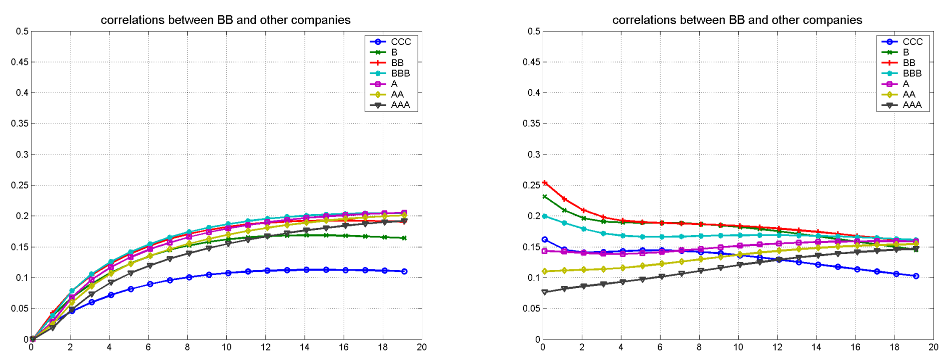

Figure 10: Correlation BB+X (b) and (c)

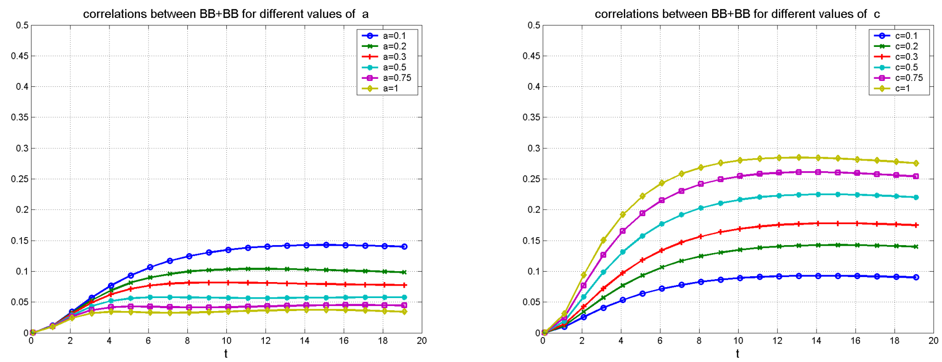

Figure 11: Correlation BB+BB for different CIR mean reversion speed and CIR volatility

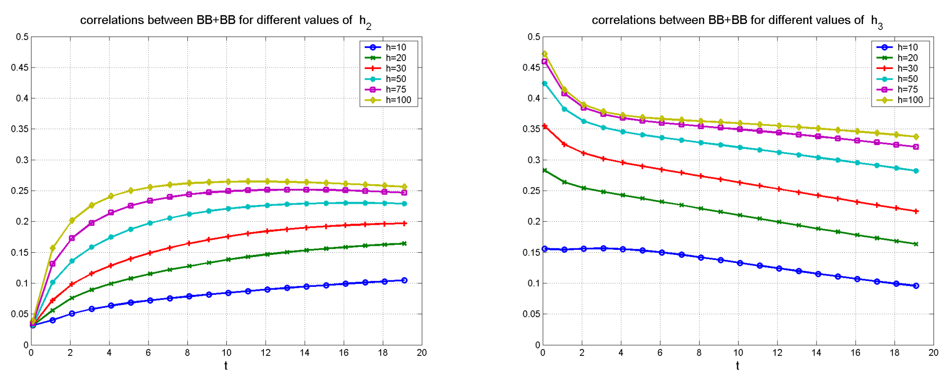

Figure 12: Correlation BB+BB for different jump sizes in Z^2 and Z^3