Fast CDO computations in the affine Markov chain model

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Abstract

Abstract: It is shown that credit basket derivatives such as CDOs which depend on a large number M of firms ($M \geq 100$ is typical in some contexts) can be modeled in a parsimonious and computationally efficient manner within the affine Markov chain (AMC) framework for multifirm credit migration introduced in a companion paper [6]. The proposed method via large M asymptotics has a number of merits. First, since our AMC models fit into the intensity based doubly stochastic framework for multifirm default, they can be flexibly fit to observed market bond data for the individual constituent firms, and they can in principle explain the dynamics of this data. Second, the method handles some of the variations of CDOs such as nonhomogeneous hazard rates and unequal notational amounts that industry practitioners need to use. Thirdly, the large M asymptotic formulas we use make sense as the first terms in an asymptotic expansion whose further terms may also prove useful in the future. Finally, in our model, prices and sensitivities for such derivatives are reduced to one dimensional integrals which can be computed on a desktop computer in seconds. In this paper we develop an illustrative version of the modeling framework and present a number of sample CDO computations which illustrate the power of the method.

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1 Introduction

The search for accurate, fast, flexible methods for pricing and hedging credit derivatives such as CDOs (collateralized debt obligations) and other large scale credit securities has become of primary importance to the finance industry in the past few years. Structured products such as these have become the paradigm of how to securitize risk in the market.

The affine Markov chain (AMC) framework introduced in [6] is an approach to the joint modeling of multifirm credit migration and default which generalizes the intensity based methods initiated in [9, 10, 4] and extended to large scale problems in [3]. We were also inspired by the affine credit default framework of [2]. The new modeling premise which underlies the framework is that firms undergo credit rating migrations which are independent conditioned on an intensity which is simply the speed of a stochastic time change. We assume that the rating classes are "ideal" in that they represent the best estimate of a firm's credit worthiness at any instant in time and that firms with the same rating have identical default risk. In the basic AMC framework adopted here, default dependence is introduced by a single stochastic time change which represents the "speed" of the credit market. This stochastic time change, together with other market factors such as the interest rate, is efficiently modeled by a multivariate affine process. Various simple components of the time change can be included which build in distinct default correlation mechanisms: a diffusive component represents the "normal market", a component in which the speed jumps leads to jumps in hazard rates, while jumps in the time itself lead to the possibility of simultaneous defaults (i.e. a "contagion" effect). The concept of a stochastic market time has been used by numerous authors to explain various models of asset prices. For example, the well known VG model [12] can be thought of as geometric Brownian motion with a gamma–distributed stochastic time change. Stochastic volatility models also have a similar interpretation. Our use of the time–change concept is in the same spirit. In [6] it was shown how essential default computations for one, two or more firms could be reduced to explicit formulas or one dimensional integrals. The present paper addresses how in the AMC setting, CDOs on M firms can be approximated using the first two terms of an asymptotic expansion in inverse powers of M developed in [7]. This approach short-circuits the "curse of dimensionality" and results in formulas which are again essentially one dimensional integrals which can be computed in a few seconds.

Our algorithms perform at speeds comparable to those of the doubly stochastic models when analytic or close–to–analytic formulas are available. Our modeling approach should also be compared to the standard industry approaches to CDOs known as copula or factor methods, of which the best known versions are those of [11, 5, 1]. These frameworks lead to algorithms for CDOs which perform at speeds comparable to ours. However, in those models hazard rates and default correlations are introduced without regard to the dynamics of the underlying companies (essentially by fitting functional forms such as a one factor normal copula to current market data). Since the models are not equivalent a direct comparison of the two modeling approaches is difficult: however, we show how the benchmark CDO computations found in [5] can be reproduced by a very simple version of our method. Our main results for CDO pricing show how a more realistic model can be computed leading to results which are qualitatively consistent with those coming from the copula/factor methods.

The organization of the paper is as follows. Section 2 introduces the main modeling ingredients for multifirm credit migration in the AMC framework, and illustrates the method by computing one-firm transition and default probabilities in terms of basic building block functions called G_1, G_2, G_3. Section 3 describes the basic structure of CDOs, the nature of the large M approximation, and how the separate components of a general CDO can be efficiently computed within this approximation. Section 4 compares the benchmark computations for the normal copula model from [5] to those of a very simple version of our model. Section 5 presents numerical results for CDO spreads in the more complex AMC setting introduced in [6] while Section 6 discusses sensitivity computations for CDOs. Finally, the formulas for the building block functions G_1, G_2, G_3 are given in an appendix.

In summary, the present paper introduces an efficient and flexible approach for evaluating prices and hedge ratios for large scale credit derivative securities such as CDOs. The results seem to pass visual inspection to be a plausible description of real credit markets.

2 The affine Markov chain model

The credit model of [6] for M firms is built from the following basic ingredients:

1. A vector of independent K–state Markov chains $\tilde{\mathbf{Y}}_t = (\tilde{Y}_t^1, \tilde{Y}_t^2, \dots, \tilde{Y}_t^M)$ where $\tilde{Y}_t^k \in \{1, \dots, K\}$ and 1 is an absorbing state.
2. The "market time" defined to be a stochastic time change process τ_t (a "subordinator");
3. The spot interest rate process r_t.

We assume that $\tilde{Y}$ is independent of r_t, τ_t. For simplicity in this paper we assume a fixed recovery fraction R_0 at default: stochastic recovery was included in [6].

The Markov chains have identical Markov generators $\mathcal{L}_Y$ (a $K \times K$ matrix) and thus identical node-to-node transition probabilities $P(\tilde{Y}_t = j | \tilde{Y}_0 = i)$ given by the semigroup $e^{t\mathcal{L}_Y}$. The market time may have absolutely continuous and jump components. Then we define the real time credit migration process to be $\mathbf{Y}_t = \tilde{\mathbf{Y}}_{\tau_t}$. The state of $Y_t^i \in \{1, 2, \dots, K\}$ represents the credit rating of firm i at time t, where the absorbing state 1 is the default state. For example, we may map these states to Standard and Poor's rating classes:

$$\{1, 2, \dots, 8\} \leftrightarrow \{\text{'default', CCC, B, BB, BBB, A, AA, AAA}\}.$$

The time of default of the ith firm is the stopping time

$$t_i^* = \inf\{t | Y_t = 1\}. \tag{1}$$

Figure 1: Ingredients of the stochastic time change

Thus the picture which describes the basic AMC model is that firms of the same rating have identical migration and default probabilities. Conditioned on r_t, τ_t, firms undergo independent credit migration with identical transition probabilities: eventually every firm defaults. The stochastic time change τ_t leads to correlations between firm defaults. When the stochastic clock is running fast, migration and hence defaults happen relatively frequently; if the stochastic clock jumps, then simultaneous defaults may occur. To create an interesting range of possibilities, we model the time change and interest rate in terms of three underlying independent factors: a two-dimensional affine process $\mathbf{Z}_t = (Z_t^1, Z_t^2)$ and a Poisson process Z_t^3. Then:

- The interest rate $r_t = \langle \mathbf{M}_r \cdot \mathbf{Z}_t \rangle$
- The time change has absolutely continuous and jump components

$$\tau_t = \tau_t^{(\mathrm{ac})} + \tau^{(\mathrm{jump})} = \int\limits_0^t \langle \mathbf{M}_\tau \cdot \mathbf{Z}_s \rangle ds + m_\tau Z_t^3. \tag{2}$$

Here $m_\tau \geq 0$ while $\mathbf{M}_r$ and $\mathbf{M}_\tau$ are coefficient vectors from $\mathbb{R}_+^2$.

To be specific, let Z^1 be a CIR process with Markov generator

$$\mathcal{L}_{Z^1} f(x) = a(1-x)f'(x) + cxf''(x), \tag{3}$$

and Z^2 be an affine process with jumps defined by its Markov generator

$$\mathcal{L}_{Z^2} f(x) = \lambda_2 (f(x+h_2) - f(x)) - h_2 \lambda_2 x f'(x). \tag{4}$$

Finally, Z^3 (a jump part of the time change) is taken to be a Poisson process with fixed jump size h_3 and intensity $\lambda_3 = h_3^{-1}$:

$$Z_t^3 = h_3 \Pi(h_3^{-1} t). \tag{5}$$

Note that Z^2 undergoes jumps of size $h_2 > 0$ with intensity λ_2 and then decays exponentially (with the speed of decay given by $h_2\lambda_2$). Also Z_t^1, Z_t^2, Z_t^3 are normalized to have long term means of $1, 1, t$ respectively so $\mathbf{M}_\tau^1 + \mathbf{M}_\tau^2 + m_\tau$ is equal to the average speed of the time change.

Remark 2.1. *Note that the M firms are exchangeable, since the components of $\tilde{\mathbf{Y}}$ are independent and firms of the same ratings class are identically distributed.*

The main computational building blocks are the functions G_1 and G_2 defined by

$$G_1(t, \mathbf{z}; \mathbf{u}, \mathbf{v}) = E_{0,\mathbf{z}} \left[e^{-\int_0^t \langle \mathbf{u} \cdot \mathbf{Z}_s \rangle ds} e^{-\langle \mathbf{v} \cdot \mathbf{Z}_t \rangle} \right] \tag{6}$$

and

$$G_2(t, \mathbf{z}; \mathbf{u}, \mathbf{v}, \mathbf{w}) = E_{0,\mathbf{z}} \left[e^{-\int_0^t \langle \mathbf{u} \cdot \mathbf{Z}_s \rangle ds} \langle \mathbf{w} \cdot \mathbf{Z}_t \rangle e^{-\langle \mathbf{v} \cdot \mathbf{Z}_t \rangle} \right]. \tag{7}$$

Here $\mathbf{z}, \mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^2$. Due to our choice of underlying factors these functions can be computed explicitly (see the appendix).

We will also need the explicit expression for the Laplace transform of Z_t^3

$$G_3(t; v) = E \left[e^{-v Z_t^3} \right] = \exp \left(h_3^{-1} t (e^{-v h_3} - 1) \right). \tag{8}$$

2.1 Transition probabilities for the process Y_t

As we showed in [6], this setup is computationally efficient: the affine structure of the stochastic time change works beautifully with the Markov chains. To illustrate, we show how one can compute node-to-node transition probabilities for each firm.

For simplicity, we assume that $\mathcal{L}_Y$ can be diagonalized (recall each component of Y has the same generator):

$$\mathcal{L}_Y = Q D Q^{-1}, \tag{9}$$

where $D = \operatorname{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_K\}$ is a diagonal matrix and $Q = (q_{ij})_{i,j=1\ldots K}$ is a matrix whose columns are the corresponding eigenvectors of $\mathcal{L}_Y$. Let the elements of Q^{-1} be denoted as $Q^{-1} = (\tilde{q}_{ij})_{i,j=1\ldots K}$.

Solving the Kolmogorov equation shows that the probability semigroup for the process Y_t is given (in matrix form)

$$\mathcal{P}_y(t) = e^{t \mathcal{L}_Y} = Q e^{tD} Q^{-1}, \tag{10}$$

and the node-to-node transition probabilities for the process $\tilde{Y}_t^i$ are given by

$$p_{yj}(t) = P_{0,y}(\tilde{Y}_t^i = j) = \sum_{k=1}^K q_{yk} \tilde{q}_{kj} e^{\alpha_k t} \tag{11}$$

Here is the result for the real time credit migration process Y_t:

Lemma 2.2. *Node to node rating transition probabilities for the ith firm are given by*

$$P_{0,\mathbf{z},y}(Y_t^i = j) = \sum_{k=1}^K q_{yk} \tilde{q}_{kj} G_1(t, \mathbf{z}; -\alpha_i \mathbf{M}_\tau, \mathbf{0}) G_3(t, -m_\tau \alpha_k). \tag{12}$$

Proof:

$$\begin{aligned} P_{0,\mathbf{z},y}(Y_t^i = j) &= E_{0,\mathbf{z},y}[I\{\tilde{Y}_{\tau_t} = j\}] = E_{0,\mathbf{z},y}[E_{0,\mathbf{z},y}[I\{\tilde{Y}_{\tau_t} = j\} | \tau]] \\ &= E_{0,\mathbf{z}} \left[\sum_{k=1}^K q_{yi} \tilde{q}_{kj} e^{\alpha_k \tau_t} \right] = \sum_{k=1}^K q_{yk} \tilde{q}_{kj} E_{0,\mathbf{z}} \left[e^{\alpha_k \tau_t} \right]. \end{aligned}$$

The result follows since

$$E_{0,\mathbf{z}} \left[e^{\alpha_i \tau_t} \right] = E_{0,\mathbf{z}} \left[e^{\alpha_i \int\limits_0^t \langle \mathbf{M}_\tau \cdot \mathbf{Z}_s \rangle ds + \alpha_i Z_t^3} \right] = G_1(t, \mathbf{z}; -\alpha_i \mathbf{M}_\tau, \mathbf{0}) G_3(t, -m_\tau \alpha_i).$$

3 Pricing large scale CDOs

Collateralized debt obligations (CDOs) are basket credit derivatives which involve a large number of companies. The underlying security is a portfolio of coupon paying corporate bonds on M firms: the ith bond is taken to have face value N_i (called the "notional"). A "synthetic CDO tranche" can be regarded as a credit swap between two parties, the insured and the insurer. The two components of the swap, the "premium leg" and the "insurance leg", are the basic credit contingent claims. We now show how each can be priced separately by risk neutral expectation. Henceforth it is assumed that all probabilities are computed in the risk neutral measure.

The components of a CDO tranche are derivatives on the total loss of the portfolio due to default of the constituent names. The loss at time t as a fraction of the total notional is given by

$$L_t = \sum_{i=1}^{M} (1 - R^i_{t^*_i}) \frac{N_i}{N} I\{t^*_i < t\} \tag{13}$$

where

- M is the number of firms;
- N_i is the notional of firm i;
- $N = \sum\limits_{i=1}^{M} N_i$ is the total notional of the portfolio;
- t^*_i is the default time of the firm i.

We make the simplifying assumption that the recovery is constant, that is $R^i_t = R_0$. Now using the structure of our model we can rewrite the total loss process as

$$L_t = \tilde{L}_{\tau_t}, \tag{14}$$

where

$$\tilde{L}_t = \sum_{i=1}^{M} (1 - R_0) \frac{N_i}{N} I\{\tilde{Y}^i_t = 1\}.$$

Thus the process $\tilde{L}_t$ is the sum of M independent random variables. Of course for large M the distribution of $\tilde{L}_t$ is very hard to compute explicitly. However since all the random variables $I\{\tilde{Y}^i_t = 1\}$ are independent, we can use the central limit theorem to approximate the distribution of $\tilde{L}_t$ as

$$\tilde{L}_t \overset{d}{\approx} L(t, \xi) = \tilde{m}(t) + \xi \tilde{\sigma}(t), \tag{15}$$

where $\tilde{m}(t)$ and $\tilde{\sigma}^2(t)$ are the mean and variance of $\tilde{L}_t$ and ξ is Gaussian $N(0, 1)$. More efficient however is the approximation

$$\tilde{L}_t \overset{d}{\approx} L(t, \xi) = [m(t) + \xi \tilde{\sigma}(t)] X_t, \tag{16}$$

where $X_t = I\{\tilde{Y}_t^i \neq 0, \text{ all i}\}$. In this case, $m(t)$ and $\sigma^2(t)$ are the mean and variance of $\tilde{L}_t|_{X_t=1}$ and can easily be computed

$$m(t) = [1 - P(t)]^{-1} \sum_{k=2}^{K} \alpha_k p_{k1}(t), \qquad \alpha_k = \sum_{i=1}^{M} (1 - R_0) I\{\tilde{Y}_0^i = k\} \frac{N_i}{N}$$

$$\sigma^2(t) = [1 - P(t)]^{-1} \sum_{k=2}^{K} \beta_k p_{k1}(t)(1 - p_{k1}(t)) - P(t) m^2(t), \qquad \beta_k = \sum_{i=1}^{M} (1 - R_0)^2 I\{\tilde{Y}_0^i = k\} \frac{N_i^2}{N^2}$$

$$P(t) = \prod_{k=2}^{K} (1 - p_{k1}(t))^{\gamma_k}, \qquad \gamma_k = \sum_{i=1}^{M} I\{\tilde{Y}_0^i = k\}$$

Provided the sequence of notionals N_i is bounded above and below uniformly in i, we note that $m(t) = O(1), \sigma(t) = O(M^{-\frac{1}{2}})$ and the approximation is accurate to $O(M^{-1})$.

The distribution of real time loss process thus is approximated by

$$L_t = \tilde{L}_{\tau_t} \overset{d}{\approx} L(\tau_t, \xi) = m(\tau_t) + \xi \sigma(\tau_t). \tag{17}$$

The paper [7] analyzes the error connected with this type of approximation.

3.1 Credit premium

A generalized premium leg for a CDO can be regarded as a contingent claim which is paid by the insured to the insurer as a fee for default insurance. It is assumed that the insured party pays continuously in time over the period $[0, T]$ at a stochastic rate U_t, and that U_t depends only on the loss process L_t:

$$U_t = U(L_t) \overset{d}{\approx} U(L(\tau_t, \xi))$$

for some function $U(x)$. A typical example is the premium leg for a CDO tranche for fractional losses in a range $[\underline{x}, \bar{x}] \subset [0, 1]$

$$U(x) = \frac{1}{\bar{x} - \underline{x}} \left[(\bar{x} - x)^+ - (\underline{x} - x)^+ \right] \tag{18}$$

Special cases are the senior tranche $\bar{x} = 1, \underline{x} > 0$, and the equity tranche $\underline{x} = 0, \bar{x} < 1$. In section 5 we consider the following standard tranches:

$$[0, 0.3], [0.03, 0.07], [0.07, 0.10], [0.10, 0.15], [0.15, 0.30], [0.30, 1.0].$$

The assumption of continuous payments in time is for simplicity of exposition only. A more pragmatic choice such as quarterly payments can be easily accommodated.

The price of the premium leg is given by

$$V^U = E_{0,\mathbf{z}} \left[\int_0^T e^{-\int_0^t r_s ds} U(L_t) dt \right] \approx V^{U*} = E_{0,\mathbf{z}} \left[\int_0^T e^{-\int_0^t r_s ds} U(L(\tau_t, \xi)) dt \right]. \tag{19}$$

Our first main result is the following formula for the approximate price of the premium leg:

Theorem 3.1. *If U is a bounded measurable function then*

$$V^{U*} = \int_0^\infty H^U(\tau) F^P(\tau;\mathbf{z}) d\tau, \tag{20}$$

where the function $H^U(\tau)$ is given by

$$H^U(\tau) = E\left[U(L(\tau,\xi))\right], \tag{21}$$

and thus depends only on the parameters of the loss process L_t and the payoff function U. The function $F^P(\tau;\mathbf{z})$ is given by

$$\begin{aligned} F^P(\tau;\mathbf{z}) &= \int_0^T E_{0,\mathbf{z}}\left[e^{-\int_0^t r_s ds}\delta(\tau_t-\tau)\right] dt \qquad (22)\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{iw\tau}\left[\int_0^T G_1(t,\mathbf{z};\mathbf{M}_r+iw\mathbf{M}_\tau,0)G_3(t,iwm_\tau)dt\right] dw, \end{aligned}$$

and thus depends only on the parameters of the interest rate and time change processes.

Proof: First note

$$V^{U*} = \int_0^T E_{0,\mathbf{z}}\left[H^U(\tau_t)e^{-\int_0^t r_s ds}\right] dt \tag{23}$$

where $H^U(\tau) = E\left[U(L(\tau,\xi))\right]$ is bounded, measurable. It is now sufficient to prove (20) for the (dense) set of complex exponential functions of the form $H^U(\tau) = e^{-iw\tau}, w \in \mathbb{R}$ with the result extending by linearity to general bounded measurable H supported on $\mathbb{R}_+$. By the Fourier inversion theorem and the basic building block formulas, we have

$$\begin{aligned} \int_0^T E_{0,\mathbf{z}}\left[e^{-iw\tau_t}e^{-\int_0^t r_s ds}\right] dt &= \int_0^T G_1(t,\mathbf{z};\mathbf{M}_r+iw\mathbf{M}_\tau,0)G_3(t,iwm_\tau)dt \\ &= \int_{-\infty}^{\infty} e^{-iw\tau}F^P(\tau;\mathbf{z})d\tau \end{aligned}$$

which completes the proof.

Remark 3.2.

1. *To compute the function $F^P(\tau;\mathbf{z})$ given by equation (29) it is sufficient to evaluate the integral in t numerically and then perform the inverse Fourier transform in w.*

2. *One nice feature of this formula is that once the function $F^P(\tau;\mathbf{z})$ has been computed (and stored) one can easily compute the prices for all tranches of a CDO simply by integrating the tranche–dependent function $H^U(\tau)$ against the tranche–independent $F^P(\tau;\mathbf{z})$. In section 3.4 we will give an explicit formula for $H^U(\tau)$.*

3. *Another important advantage is that our formula for the price of the premium leg separates the effects of the stochastic time change (hidden in $F^P(\tau;\mathbf{z})$) from all information about the Markov chains $\tilde{\mathbf{Y}}$, the loss process L_t and the payoff function U (hidden in $H^U(\tau)$).*

3.2 Credit insurance

The insurer pays the insured a tranche of the losses by default of firms in the basket portfolio. A general claim of this type is defined by

$$W^S = E_{0,\mathbf{z}}\left[\int_0^T e^{-\int_0^t r_s ds} dS_t\right] \tag{24}$$

where $S_t = S(L_t)$ for some deterministic function $S(L)$ with $S(0) = 0$. A typical example is the default leg of a CDO tranche with range $[\underline{x}, \bar{x}]$ where:

$$S(x) = \frac{1}{\bar{x} - \underline{x}}\left[(x - \underline{x})^+ - (x - \bar{x})^+\right] = 1 - U(x) \tag{25}$$

First we simplify the expression for W^S by integrating by parts ($S(t)$ is of finite variation)

$$W^S = E_{0,\mathbf{z}}\left[e^{-\int_0^T r_s ds} S_T + \int_0^T r_t e^{-\int_0^t r_s ds} S_t dt\right], \tag{26}$$

Since $S(L_t) \overset{d}{\approx} S(L(\tau_t, \xi)$ we can approximate the price of the insurance leg as

$$W^S \approx W^{S*} = E_{0,\mathbf{z}}\left[e^{-\int_0^T r_s ds} S(L(\tau_T, \xi)) + \int_0^T r_t e^{-\int_0^t r_s ds} S(L(\tau_t, \xi)) dt\right]. \tag{27}$$

The analog of theorem 3.1 for the approximate price of the insurance leg is:

Theorem 3.3. *If S is bounded and measurable with $S(0) = 0$ then*

$$W^{S*} = \int_0^\infty H^S(\tau) F^I(\tau, \mathbf{z}) d\tau, \tag{28}$$

where $H^S(\tau)$ is given by

$$H^S(\tau) = E\left[S(L(\tau, \xi))\right].$$

The function $F^I(\tau;\mathbf{z})$ depends only on the parameters of the interest rate and time change processes and is given by

$$\begin{aligned} F^I(\tau;\mathbf{z}) \quad = \quad & \frac{1}{2\pi}\int_{\mathbb{R}} e^{iw\tau}\left\{\int_0^T G_2(t,\mathbf{z};\mathbf{M}_r + iw\mathbf{M}_\tau, 0, \mathbf{M}_r) G_3(t, iwm_\tau) dt\right. \\ & \left. + G_1(T,\mathbf{z};\mathbf{M}_r + iw\mathbf{M}_\tau, 0) G_3(T, iwm_\tau)\right\} dw. \end{aligned} \tag{29}$$

Figure 2: Computing the insurance leg: integrating $H^S(\tau)$ against $F^I(\tau, \mathbf{z})$

Figure 3: Functions $H^S(\tau)$ for $M = 100$(blue) and $M = \infty$(red). The x-axis is total loss L

3.3 CDO tranche spreads

A simple CDO tranche $[\underline{x}, \bar{x}] \subset [0, 1]$ is a swap (a contract with zero value at time 0) of a multiple of the premium leg V^U with U given by (18) for a default leg W^S with S given by (25). The "spread" $p = W^S/V^U$ is selected to balance the two legs.

3.4 Computing the H functions

The following proposition gives an explicit formula for $H^S(\tau) = E[S(L(\tau, \xi)]$ when S has the form (25). The other important function is $H^U(\tau) = 1 - H^S(\tau)$.

Proposition 3.4.

$$H^S(\tau) = \frac{\sigma(\tau)}{\bar{x} - \underline{x}} \left(E[(\xi - \underline{d}(\tau))^+] - E[(\xi - \bar{d}(\tau))^+] \right) \tag{30}$$

where $\underline{d}(\tau) = \frac{\underline{x} - m(\tau)}{\sigma(\tau)}$ *and* $\bar{d}(\tau) = \frac{\bar{x} - m(\tau)}{\sigma(\tau)}$.

The proof is easy. Straightforward integration by parts also shows that

$$E(\xi - A)^+ = A(\Phi(A) - 1) + \frac{1}{\sqrt{2\pi}} e^{-\frac{A^2}{2}}, \tag{31}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$ is the cumulative distribution function of $N(0, 1)$.

4 Comparison with the normal copula model

To compare our model with the one factor normal copula model (see [5]) we use the following modeling assumptions: the Markov chains $\tilde{Y}_t^i$ are just two-state ($K = 2$) processes with Markov generator $\mathcal{L}_Y$ given by

$$\mathcal{L}_Y = \begin{pmatrix} 0 & 0 \\ \lambda & -\lambda \end{pmatrix},$$

where the default intensity has the value $\lambda = 0.01$. Other parameters are: a constant recovery rate $R_0 = 0.4$; a constant interest rate $r_t = 0.05$.

The important difference between the two models is the correlation structure. While the copula model has a unique parameter ρ which explains all correlations between default events, in our model we have several parameters which are responsible for the default correlations (the most important ones are h_2, h_3, λ_2). In order to compare the correlation structures of the two models we look at the term structure of default correlations given by the following function

$$\mathrm{corr}_{ij}(t) = \mathrm{corr}(I\{t_i^* < t\}, I\{t_j^* < t\}) \tag{32}$$

where "corr" is the usual correlation between random variables. To match the correlation term structures we simply choose parameters of the time change process which give a reasonable fit. For a detailed discussion of the correlation issues see [6].

tranche	$\rho = 0.1$, HW	$\rho = 0.1$, AMC	$\rho = 0.3$, HW	$\rho = 0.3$, AMC
[0,0.03]	2279	2221	1487	1359
[0.03,0.06]	450	469	472	425
[0.06,0.10]	89	86	203	236
[0.10,1.00]	1	0.5	7	6

We observe that although these two models are certainly different, they produce qualitatively similar results across all tranches.

5 CDOs in a toy model

In this section we apply the model with $K - 1 = 7$ rating classes (these can be interpreted as an idealization of the Moody's or Standard and Poor's system). The first step is to specify the Markov generator $\mathcal{L}_Y$. The relation between historical and risk neutral transition probabilities is discussed in [8]. Here we simply take $\mathcal{L}_Y = 2 \times \mathcal{L}_{\mathrm{Hist}}$, where the factor 2 represents the market default risk premium:

$$\mathcal{L}_Y = 2 \times \begin{pmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.2856 & -0.4318 & 0.0928 & 0.0250 & 0.0142 & 0.0142 & 0.0000 & 0.0000 \\ 0.0753 & 0.0479 & -0.1928 & 0.0568 & 0.0073 & 0.0034 & 0.0021 & 0.0000 \\ 0.0273 & 0.0144 & 0.1181 & -0.2530 & 0.0813 & 0.0089 & 0.0025 & 0.0005 \\ 0.0049 & 0.0020 & 0.0174 & 0.0701 & -0.1711 & 0.0713 & 0.0047 & 0.0007 \\ 0.0010 & 0.0000 & 0.0048 & 0.0107 & 0.0688 & -0.1172 & 0.0309 & 0.0010 \\ 0.0000 & 0.0000 & 0.0030 & 0.0030 & 0.0105 & 0.0787 & -0.1043 & 0.0091 \\ 0.0000 & 0.0000 & 0.0000 & 0.0031 & 0.0020 & 0.0083 & 0.1019 & -0.1153 \end{pmatrix}$$

The matrix $\mathcal{L}_{\mathrm{Hist}}$ was taken from [8].

We use the two sets of parameters chosen in [6]. In that paper, the parameters were chosen to yield credit spread curves and correlation curves which were quantitatively similar to the benchmark problems studied by [5, 11]. The "$\rho = 0.1$ model" is defined to have the parameters

$$a = c = 0.1, \quad h_2 = 3, \lambda_2 = 0.3, \quad h_3 = 3,$$

Figure 4: Credit Spreads (hazard rates)

and is quantitatively similar to the one–factor Gaussian copula model with correlation $\rho = 0.1$ The "$\rho = 0.3$ model" is defined to have the parameters

$$a = c = 0.1, \quad h_2 = 15, \lambda_2 = 0.05, \quad h_3 = 5,$$

and is quantitatively similar to the one–factor Gaussian copula model with correlation $\rho = 0.3$. In both cases the marginal default time distribution is the same as in our model. In all models we take the fractional recovery to have the fixed value $R_0 = 0.4$, parameters for the interest rate are fixed to be $\mathbf{M}_r = (0.05, 0)$ and the parameters for the time change are $\mathbf{M}_\tau = (0.3, 0.6)$ and $m_\tau = 0.1$. The Z process is initialized with $(Z_0^1, Z_0^2) = (1, 1)$.

In each model, we consider the six standard tranches for two typical 5 year CDOs. Contract A has M similar firms, divided equally into the four investment grade ratings classes (BBB, A, AA, AAA); while contract B has all the M companies in the BBB rating class. In both cases the CDO is structured with equal nominal values.

When combined with the matrix $\mathcal{L}_Y$ above the model for individual firms leads as in [6] to credit default spread curves for each rating class as shown in Figure 5. The following table shows the tranche spreads which result for the four contracts described above:

tranche	$\rho = 0.1$, A	$\rho = 0.1$,B	$\rho = 0.3$,A	$\rho = 0.3$,B
[0,0.03]	1193	2538	876	1634
[0.03,0.07]	113	692	302	537
[0.07,0.10]	7	187	130	383
[0.10,0.15]	0.3	37	30	262
[0.15,0.30]	0	1	1	46
[0.30,1.00]	0	0	0	0.1

The computations on this table took 2.8 seconds when implemented in MATLAB on a 2.4GHz laptop.

6 Sensitivity

One good thing about an analytical or semi–analytical treatment of securities such as ours compared to Monte Carlo based methods is that sensitivity analysis is both conceptually and computationally straightforward. In our model, security prices are sensitive to the underlying dynamic risk factors $\tilde{Y}, Z^1, Z^2, Z^3$: these evolve in time, so they are the most important factors for risk management. Next in importance are the model parameters which are taken to be constant in time, but are subject to calibration error: these are $(\mathcal{L}_Y, a, c, \lambda_2, h_2, h_3, \mathbf{M}_r, \mathbf{M}_\tau, m_\tau)$. In the present discussion we focus on hedging the sensitivities to the dynamic factors, and leave parameter hedging for future study.

Since Z^1, Z^2 control the overall shape of credit spread and correlation curves, hedging these factors may be thought of as hedging general market risk. The most important hedge is thus to create delta–neutral combinations with respect to these risk factors. Fortunately, the requisite derivatives of both the premium and insurance legs are explicitly computable:

$$(\Delta_{V,1}, \Delta_{V,2}) = \partial_{\mathbf{z}} V^U = \int_0^\infty H^U(\tau) \partial_{\mathbf{z}} F^P(\tau, \mathbf{z}) d\tau, \tag{33}$$

$$(\Delta_{W,1}, \Delta_{W,2}) = \partial_{\mathbf{z}} W^S = \int_0^\infty H^S(\tau) \partial_{\mathbf{z}} F^I(\tau, \mathbf{z}) d\tau, \tag{34}$$

where $\partial_{\mathbf{z}} F^P, \partial_{\mathbf{z}} F^I$ are explicit in terms of the building blocks defined so far. Thus hedging general market risk is a tractable problem in our model. Figures 5,6,7,8 show graphs of V^U, W^S against Z_0^1, Z_0^2.

Next we consider hedging the risk factors Y. This amounts to protecting against the risk of any individual downgrade, upgrade or default, and such firm specific risks can only be delta hedged by holding additional credit securities on each name of the basket. Similarly hedging for jumps of Z^3 involves firm specific risks. For large scale baskets this type of hedging is of secondary importance.

7 Conclusion

The AMC framework gives dynamical models of multifirm credit migration and default which fall in the class of reduced form or doubly stochastic models. The particular way of combining a continuous time Markov chain with an independent set of affine processes yields a flexible framework within which computations are very efficient.

In this paper and its companion [6] we have demonstrated methods for computing credit spreads, correlation curves and CDO tranches, all of which pass visual inspection to be a plausible representation of real markets. In contrast to typical static copula models for large scale basket derivatives, our approach attempts to capture a resemblance to real market dynamics.

The increased realism of our framework does not lead to slower computation times. In fact, in our specification, the computation times are not particularly longer than what can be achieved in a simple one factor normal copula CDO model. In common with such models, the loss process is a sum over firms of conditionally independent random variables. In contrast, however, the conditioning variable in our modeling is a stochastic process τ_t rather than a single random variable. Nonetheless, the computation speeds are similar because the affine structure of the conditioning process τ_t reduces critical computations to one–dimensional integrals.

The version we have presented appears to be an example of a rather general dynamic approach to credit risk, one which can be extended in several distinct directions. We mention here two important types of improvements which are easy to add. The first is to replace the conditioning process τ_t by m-dimensional processes which might include for example the stochastic recovery or a variety of time changes for different sectors of the economy. It appears that our CDO pricing framework extends easily to this setting and would yield formulas for CDOs involving m–dimensional integrals and Fourier transforms. A type of extension along these lines which is not in the spirit of our approach to CDOs would be to include idiosyncratic risk factors which are specific to individual firms: these would lead to a curse of dimensionality with integrals in M dimensions.

We have shown that our approximate CDO prices have M dependence consistent with an $O(1/M)$ error. Another direction for improvement of the method would be to improve the large M asymptotics of the errors $|V^U - V^{U*}|$, $|W^S - W^{S*}|$ by including further correction terms. [7] derives a very general asymptotic expansion which can perform this task.

Our aim here has been to demonstrate that the AMC framework is flexible enough in principle to fit defaultable bond data and credit derivatives such as CDOs. A detailed study of the validity of the approach for modeling real data sets is clearly justified.

A Formulas for main building blocks

For the reader's convenience, we reproduce here computations given in 04a.

A.1 Computing G_1

Since Z^1, Z^2 are independent, we have

$$\begin{aligned} G_1(t, \mathbf{z}; \mathbf{u}, \mathbf{v}) &= E_{0,\mathbf{z}}\left[e^{-\int_0^t \langle \mathbf{u} \cdot \mathbf{Z}_s \rangle ds} e^{-\langle \mathbf{v} \cdot \mathbf{Z}_t \rangle}\right] \\ &= \prod_{i=1}^2 E_{0,z^i}\left[e^{-u^i \int_0^t Z_s^i ds} e^{-v^i Z_t^i}\right] = \prod_{i=1}^2 G_1^{(Z^i)}(t, z^i; u^i, v^i) \end{aligned}$$

Since Z^1 is a CIR process function $G_1^{(Z^1)}$ is well known. Let Z be a CIR process

$$dZ_t = (a - bZ_t)dt + c\sqrt{Z_t} dW_t.$$

Then $G_1^{(Z)}$ is given by

$$G_1^{(Z)}(t, z; u, v) = E_{0,z}\left[e^{-u \int_0^t Z_s ds} e^{-v Z_t}\right] = e^{\phi(t,u,v) + z\psi(t,u,v)}, \tag{35}$$

and functions ϕ and ψ are computed as

$$\begin{cases} \phi(t,u,v) = a\psi_2 t - \frac{a}{c}\log\left(e^{-\gamma t} + \frac{c}{\gamma}(v+\psi_1)\left(1-e^{-\gamma t}\right)\right) \\ \psi(t,u,v) = \psi_2 - \frac{v+\psi_2}{1+\frac{c}{\gamma}(v+\psi_1)(e^{\gamma t}-1)}, \end{cases} \tag{36}$$

and constants ψ_1, ψ_2 and γ are given by

$$\begin{cases} \gamma = \sqrt{b^2+4uc} \\ \psi_1 = \frac{b+\gamma}{2c} \\ \psi_2 = \frac{b-\gamma}{2c} \end{cases} \tag{37}$$

Process $Z = Z^2$ with the Markov generator

$$\mathcal{L}_Z f(x) = \Lambda(f(x+h) - f(x)) - bxf'(x)$$

is again affine. Thus it's function $G_1^{(Z)}$ is computed as

$$G_1^{(Z)}(t,z;u,v) = E_{0,z}\left[e^{-u\int_0^t Z_s ds} e^{-vZ_t}\right] = e^{\phi(t,u,v)+z\psi(t,u,v)}, \tag{38}$$

and functions ϕ and ψ are given as solutions to the following system of equations

$$\begin{cases} \frac{d\psi}{dt} = -b\psi - u, \quad \psi(0,u,v) = -v \\ \frac{d\phi}{dt} = \Lambda(e^{h\psi} - 1), \quad \phi(0,u,v) = 0. \end{cases} \tag{39}$$

This system can be solved explicitly to give the following expressions

$$\begin{cases} \psi(t,u,v) = \left(\frac{u}{b} - v\right)e^{-bt} - \frac{u}{b} \\ \phi(t,u,v) = \frac{\Lambda}{b} e^{-\frac{uh}{b}} (Ei(h(\frac{u}{b}-v)) - Ei(h(\frac{u}{b}-v)e^{-bt})) - \Lambda t, \end{cases} \tag{40}$$

where $Ei(x)$ is the special function called *exponential integral* and defined as a Cauchy principal value of the integral $\int_{-\infty}^{x} e^y/y dy$.

A.2 Computing G_2

Here again we can use the fact that processes Z^1, Z^2, Z^3 are independent to simplify function G_2 as

$$G_2(t,\mathbf{z};\mathbf{u},\mathbf{v},\mathbf{w}) = E_{0,\mathbf{z}}\left[e^{-\int_0^t \langle \mathbf{u}\cdot\mathbf{Z}_s\rangle ds} \langle \mathbf{w}\cdot\mathbf{Z}_t\rangle e^{-\langle \mathbf{v}\cdot\mathbf{Z}_t\rangle}\right]$$
$$= w^1 G_2^{(Z^1)}(t,z^1;u^1,v^1) G_1^{(Z^2)}(t,z^2;u^2,v^2) + w^2 G_1^{(Z^1)}(t,z^1;u^1,v^1) G_2^{(Z^2)}(t,z^2;u^2,v^2).$$

Here for a process Z function $G_2^{(Z)}(t,z;u,v)$ is defined as

$$\begin{aligned} G_2^{(Z)}(t,z;u,v) &= E_{0,z}\left[e^{-\int_0^t uZ_s ds} Z_t e^{-vZ_t}\right] \\ &= -\frac{\partial}{\partial v} E_{0,z}\left[e^{-\int_0^t uZ_s ds} e^{-vZ_t}\right] = -\frac{\partial}{\partial v} G_1^{(Z)}(t,z;u,v). \end{aligned}$$

Since we have explicit expression for individual functions G_1 we can easily find explicit formulas for G_2:

$$\begin{aligned} G_2^{(Z)}(t,z;u,v) &= -\frac{\partial}{\partial v} G_1^{(Z)}(t,z;u,v) = -\frac{\partial}{\partial v} e^{\phi(t,u,v)+z\psi(t,u,v)} \\ &= -\left(\frac{\partial}{\partial v}\phi(t,u,v) + x\frac{\partial}{\partial v}\psi(t,u,v)\right) G_1^{(Z)}(t,z;u,v). \end{aligned}$$

Thus to compute individual functions G_2 we just need to compute the partial derivative in v of ϕ and ψ. In the case of CIR process they can be easily found from the formula (36):

$$\begin{cases} \frac{\partial}{\partial v}\psi(t,u,v) = -\frac{e^{\gamma t}}{1+\frac{c}{\gamma}(v+\psi_1)(e^{\gamma t}-1)} \\ \frac{\partial}{\partial v}\phi(t,u,v) = -\frac{a}{\gamma}\frac{e^{\gamma t}-1}{1+\frac{c}{\gamma}(v+\psi_1)(e^{\gamma t}-1)} \end{cases} \tag{41}$$

For the process Z^2 these partial derivatives are easily found from the formula (40)

$$\begin{cases} \frac{\partial}{\partial v}\psi(t,u,v) = -e^{-bt} \\ \frac{\partial}{\partial v}\phi(t,u,v) = -\frac{\Lambda}{u-bv}e^{-\frac{uh}{b}}\left(e^{h(\frac{u}{b}-v)} - e^{h(\frac{u}{b}-v)e^{-bt}}\right) \end{cases} \tag{42}$$

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Figure 5: the dependence of premium leg on the initial value Z_0^1

Figure 6: the dependence of premium leg on the initial value Z_0^2

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Figure 7: the dependence of insurance leg on the initial value Z_0^1

Figure 8: the dependence of insurance leg on the initial value Z_0^2

Figure 9: the dependence of CDO spreads on the initial value Z_0^1

Figure 10: the dependence of CDO spreads on the initial value Z_0^2

Figure 11: The dependence of CDO spreads on the jumpsize h_2

Figure 12: The dependence of CDO spreads on the jumpsize h_3