ECO 317 – Fall Term 2009
Economics of Uncertainty
Final Examination
Answer Key

Here is some statistical information. On the 180-point basis, for the whole exam the Median was 104 (58%), Mean 107.3 (59.4%), Standard deviation 19.3 (10.75%).

Distribution by percentage

Range	Number
90–99	0
80–89	0
70–79	1
60–69	6
50–59	7
0–49	1

Average by question

Question	Average
1	21.3/25
2	22.8/30
3	12.8/30
4	11.8/30
5	17.3/30
6	21.3/35

GENERAL COMMENT:

Performance on the math questions 3–5 was generally very disappointing. We have been quite generous in grading and letter-grade assignment, but I had hoped for better.

Question 1: (25 points)

No standard answer can be provided. You can base your answer on Lecture Note 10, EGS textbook Chapter 13, and other readings such as Machina. And for the "reasons" part you have to supply some original thinking :-(. Your answers are judged on (a) the clarity and correctness of your brief description of the chosen theories, (b) the cogency of your reasoning for regarding one as more damaging than the other. All this is a judgment call.

COMMENT: The answers were generally quite good. A few statements of the critiques were too skimpy and cryptic; some diagrams were poorly labeled. I deliberately left it to you to interpret "more damaging". Probably the best sense is "requiring a more drastic or more fundamental alteration to the theory," and most people so interpreted it.

Question 2: (30 points)

COMMENT: Too many of you blithely assumed that Safer and Riskier are independent. They are not, even though FF, GG and HH are mutually independent. This is because both Safer and Riskier contain FF as a common component. See the calculation below. When the error was carried over into Part (b) and the calculation was correct thereafter, I have been reasonably generous (usually 7 points out of 10).

There are two ways to do part (a). The first is easier, but the second is equally valid.

(a1) (6 points) Your portfolio is equivalent to holding 0.5 of FF, $0.5\,x$ of GG, and $0.5\,(1-x)$ of HH. Therefore (6 points)

$$\mathbf{E}[W] = 1.1 \times 0.5 + 1.2 \times 0.5\,x + 1.3 \times 0.5\,(1-x) = 1.2 + 0.6\,x - 0.65\,x = 1.2 - 0.05\,x$$

and (8 points) as these three are independent,

$$\begin{aligned}\mathbf{V}[W] &= 0.01 \times 0.25 + 0.04 \times 0.25\,x^2 + 0.09 \times 0.25\,(1-x)^2 \\ &= 0.0250 - 0.045\ x + 0.0325\ x^2\,.\end{aligned}$$

OR

(a2) (8 points) The expected rate of return on Safer is $\frac{1}{2}\,(1.1+1.2) = 1.15$ and its variance is

$$\tfrac{1}{4}\ [(0.1)^2 + (0.2)^2] = \tfrac{1}{4}\ 0.05 = 0.0125\,.$$

The expected rate of return on Riskier is $\frac{1}{2}\,(1.1+1.3) = 1.20$ and its variance is

$$\tfrac{1}{4}\ [(0.1)^2 + (0.3)^2] = \tfrac{1}{4}\ 0.10 = 0.0250\,.$$

(7 points) Moreover, because of the common presence of FF in the two funds, their returns have positive covariance. If we write the deviation of the rates of return from the mean on the three stocks as f, g and h, the deviations for Safer and Riskier are respectively $\frac{1}{2}\,(f+g)$ and $\frac{1}{2}\,(f+h)$, so the covariance is

$$\mathbf{E}[\tfrac{1}{4}\,(f+g)(f+h)] = \tfrac{1}{4}\,\mathbf{E}[f^2] = \tfrac{1}{4}\,\mathbf{V}[f] = \tfrac{1}{4}\,0.01 = 0.0025\,,$$

Since f, g and h are independent, $\mathbf{E}[fg] = 0$ etc.

(5 points) Therefore

$$\mathbf{E}[W] = 1.15\ x + 1.2\ (1-x) = 1.2 - 0.05\,x,$$

and

$$\begin{aligned}\mathbf{V}[W] &= 0.0125\ x^2 + 2 \times 0.0025\ x(1-x) + 0.0250\ (1-x)^2 \\ &= 0.0250 - 0.045\ x + 0.0325\ x^2\end{aligned}$$

(b) (10 points) To maximize $\mathbf{E}[W] - 4\ \mathbf{V}[W]$, the first-order condition is

$$-0.05 - 4\ [-0.045 + 0.065\,x] = 0\,,$$

or

$$0.13 - 0.26\ x = 0\,,$$

or

$$x = 0.5\,.$$

So you should invest 50 percent of your initial wealth in Safer, and 50 percent in Riskier.

If you followed method (a2), you could also use the formula for allocation between two risky assets, p. 5 of the Lecture 6 slides. There the objective function is $\mathbf{E}[W] - \frac{1}{2}\,\alpha\,\mathbf{V}[W]$ (see bottom of p. 1 of that handout), so $\alpha = 8$, not 4.

Question 3: (30 points)

COMMENTS:

(1) Many assumed $p_1 + p_2 = 1$ from the very beginning and so the got determinate prices.

(2) In part (b) only a few mentioned interest rate as a factor that may determine absolute values of ADS.

(3) In part (c) few translated the final consumprtion into the betting behavior.

(a) Let p_1 denote the price of an Arrow-Debreu security (ADS) that pays \$1 in the event of a Democratic victory and 0 in the event of a Republican victory, and p_2 the price of each ADS that pays \$1 in the event of a Republican victory and 0 in the event of a Democratic victory. Let the wealth of each Democrat be 1 and that of each Republican 2 by choice of units. Then each Democrat initially owns 1 of each ADS and each Republican owns 2 of each ADS. Write W_j^i for the number of ADS of type j chosen by a supporter of party i.

(12 points) Each Democrat's expected utility is

$$0.7 \ln(W_1^d) + 0.3 \ \ln(W_2^d) \, ,$$

and this is maximized subject to the budget constraint

$$p_1 \ W_1^d + p_2 \ W_2^d = p_1 + p_2 \, .$$

The usual Cobb-Douglas maximization yields

$$W_1^d = 0.7 \ \frac{p_1 + p_2}{p_1} \, , \qquad W_2^d = 0.3 \ \frac{p_1 + p_2}{p_2} \, .$$

Similarly for each Republican,

$$W_1^r = 0.4 \ \frac{2 \, (p_1 + p_2)}{p_1} \, , \qquad W_2^r = 0.6 \ \frac{2 \, (p_1 + p_2)}{p_2} \, .$$

(10 points) Suppose there are N Democrats and N Republicans. Then the equilibrium condition in the market for the Democrat ADS is

$$N \ W_1^d + N \ W_1^r = N + 2 \, N \qquad \text{or} \qquad W_1^d + W_1^r = 3 \, ,$$

or

$$\frac{(0.7 + 2 \times 0.4) \, (p_1 + p_2)}{p_1} = 3, \qquad \text{or} \qquad p_1 = 0.5 \, (p_1 + p_2) \, ,$$

or

$$p_2 = p_1 \, .$$

(b) (3 points) The absolute prices are not determinate with the information given, as usual in general equilibrium because of homogeneity and Walras' law. They could become determinate if other information e.g. about intertemporal behavior – saving and interest rates – were given.

(c) (5 points) Using these prices, we find

$$W_1^d = 1.4,\ W_2^d = 0.6, \qquad W_1^r = 1.6,\ W_2^r = 2.4\,.$$

In other words, if each Democrat has \$1,000, he makes a bet that will win him \$400 in the event of a Democratic victory and lose \$400 in the event of a Republican victory, and each Republican makes the opposite bet.

Question 4: (30 points)

COMMENTS:

(1) Several people assumed $U(Q-W) = Q-W$. This is not so; the question states that the owner is risk-averse and that U is strictly concave.

(2) Several people did not realize that when E is verifiable, the contract can specify E and therefore you need a first-order condition with respect to it.

(a) (4 points) Your expected utility

$$EU_o = F(E)\ U(Q_1 - W_1) + [1 - F(E)]\ U(Q_2 - W_2)\,.$$

The manager's expected utility

$$EU_m = F(E)\ W_1 + [1 - F(E)]\ W_2 - k\ E\,.$$

(b) (15 points) When effort is verifiable, your contract can specify all three variables (E, W_1, W_2). You will choose these to maximize EU_0 subject to $EU_m \geq u_m$. The Lagrangian is

$$L = F(E)\ U(Q_1 - W_1) + [1 - F(E)]\ U(Q_2 - W_2) + \lambda\ \{\ F(E)\ W_1 + [1 - F(E)]\ W_2 - k\ E - u_m\ \}\,.$$

The first-order conditions are

$$\frac{\partial L}{\partial W_1} = F(E)\ [\ \lambda - U'(Q_1 - W_1)\ ] = 0 \tag{1}$$

$$\frac{\partial L}{\partial W_2} = [1 - F(E)]\ [\ \lambda - U'(Q_2 - W_2)\ ] = 0 \tag{2}$$

$$\frac{\partial L}{\partial E} = F'(E)\ [\ U(Q_1 - W_1) - U(Q_2 - W_2) + \lambda\,(W_1 - W_2)\ ] - \lambda\ k = 0 \tag{3}$$

From (1) and (2) we get

$$U'(Q_1 - W_1) = U'(Q_2 - W_2) = \lambda\,.$$

Since $U'' < 0$, this implies

$$Q_1 - W_1 = Q_2 - W_2 \tag{4}$$

and then

$$U(Q_1 - W_1) = U(Q_2 - W_2)\,.$$

Substituting in (3), we get

$$F'(E)\ (W_1 - W_2) = k\,.) \tag{5}$$

(Strictly speaking we have to prove $\lambda \neq 0$, but that is easy so long as U is strictly increasing; you were not expected to spot or do this. 2 point bonus to those who did.)

Now (4) implies $W_1 - W_2 = Q_1 - Q_2$. Therefore (5) becomes

$$F'(E)\ (Q_1 - Q_2) = k\,. \tag{6}$$

Then the solution is easy to complete. (Let $Q_1 - W_1 = Q_2 - W_1 = C$, substitute into the participation constraint to solve for C, recognizing the fact that (5) defines E in terms of the exogenous variables Q_1, Q_2, k. Points not deducted if you do not state all this process of completing the solution.

(c) (6 points) When effort is unverifiable, your contract can only specify W_1 and W_2. The manager chooses E to maximize EU_m, yielding the first-order condition

$$F'(E)\ (W_1 - W_2) = k\,.$$

But this is exactly the E-FOC of your optimal contract when you could specify E also. Therefore you can implement that optimum even in the present more constrained context!

(d) (5 points) The key is that the manager is risk-neutral. Therefore it is optimal in the first-best (effort verifable) to place all the risk on him. That is what (4) shows: your net receipts in the two states of the world are equal. Then, even when effort is unverifiable, the manager can be given incentive of full power to achieve the first best effort. This is equivalent to selling the firm to the manager in return for a fixed fee.

Question 5: (30 points)

COMMENTS:

(1) A lot of student didn't explain in (d) why the two conditions are bounding. This could've been obvious, 4 points off is nothing at all is mentioned.

(2) Only a few got the right intuition in (e); given how much time we had spent on this in class very recently, we were pretty strict about it.

(3) Many forgot to compare Q_1, Q_2, so their comparison in (f) wasn't fully rigorous.

(a) (2 points) The participation constraints are $R_i - c_i\,Q_i \geq 0$ for $i = 1,\,2$.

(b) (2 points) The incentive compatibility constraints for the two types are

$$\begin{array}{ll} \text{Type 1:} & R_1 - c_1\,Q_1 \geq R_2 - c_1\,Q_2 \\ \text{Type 2:} & R_2 - c_2\,Q_2 \geq R_1 - c_2\,Q_1 \end{array}$$

(c) (1 point) The expression for the monopsonist's expected profit is

$$\mathbf{E}[\Pi] = \theta_1\ [B(Q_1) - R_1] + \theta_2\ [B(Q_2) - R_2]\,.$$

(d) (8 points up to expressions for R_1, R_2 below) Assuming that the participation constraint of type 1 and the incentive compatibility constraint of type 2 can be ignored, the remaining constraints are the participation constraint of type 2 and the incentive compatibility constraint of type 1

$$\begin{aligned} \text{PC}_2: \quad R_2 - c_2\,Q_2 &\geq 0\,, \\ \text{IC}_1: \quad R_1 - c_1\,Q_1 &\geq R_2 - c_1\,Q_2\,. \end{aligned}$$

R_2 and R_1 both enter negatively in the objective function. Lowering R_2 raises the objective function and relaxes the IC_1 constraint. Therefore this should be done until PC_2 is hit. Then R_1 should be lowered until IC_1 is hit. So both constraints should be met with equality:

$$\begin{aligned} R_2 &= c_2\,Q_2\,, \\ R_1 &= c_1\,Q_1 + R_2 - c_1\,Q_2 = c_1\,Q_1 + (c_2 - c_1)\,Q_2\,. \end{aligned}$$

(5 points for the rest of part d) Substituting this into the objective function, it becomes

$$\mathbf{E}[\Pi] = \theta_1\,[B(Q_1) - c_1\,Q_1 - (c_2 - c_1)\,Q_2] + \theta_2\,[B(Q_2) - c_2\,Q_2]\,.$$

Then the first-order conditions are

$$\begin{aligned} \frac{\partial \mathbf{E}[\Pi]}{\partial Q_1} &= \theta_1\,[B'(Q_1) - c_1] = 0 \\ \frac{\partial \mathbf{E}[\Pi]}{\partial Q_2} &= -\theta_1\,(c_2 - c_1) + \theta_2\,[B'(Q_2) - c_2] = 0\,, \end{aligned}$$

Second-order conditions are satisfied because of strict concavity ($B'' < 0$), and the property $B'(0) = \infty$ rules out the corner solutions at $Q_1 = 0$ or $Q_2 = 0$. (**** Andrei: take only 1 point off for missing one or both of these issues.)

(e) (2 points) Solving these immediately yields the results

$$B'(Q_1) = c_1, \qquad B'(Q_2) = c_2 + \frac{\theta_1}{\theta_2}\,(c_2 - c_1)\,.$$

(4 points) The intuition is that type 1 must be given enough rent to overcome its temptation to pretend to have the higher cost c_2, and then to economize on the rent loss the quantity bought from type 2 is distorted downward. But no distortion is needed on the quantity bought from type 1 because there is no even lower cost type with a temptation to pretend to have cost c_1.

(f) (6 points) At the optimum,

$$R_1 = c_1\,Q_1 + (c_2 - c_1)\,Q_2 \geq c_1\,Q_1\,,$$

so the participation constraint of type 1 is met (and with slack if $Q_2 > 0$). Also,

$$R_1 - c_2\,Q_1 = c_1\,Q_1 + (c_2 - c_1)\,Q_2 - c_2\,Q_1 = -(c_2 - c_1)\,(Q_1 - Q_2)\,.$$

But

$$B'(Q_2) > c_2 > c_1 = B'(Q_1)\,,$$

and $B'' < 0$ so $Q_1 > Q_2$. Therefore

$$R_1 - c_2\,Q_1 < 0 = R_2 - c_2\,Q_2\,,$$

so the incentive compatibility constraint of type 2 is met.

Question 6: (35 points)

This is based on your reading and discussion for the final two precepts. No answer key can be provided because there are too many possible variants for your choice of the problem and the discussion.

The plan was that a generally good and correct answer should get 28-30 points. One that is in addition well-organized and well-written can go to 32-33; only a really exceptional answer, better than one we could have written, would get 35. An answer with generally poor quality but no serious errors can go down to 20-22. Serious errors can lower it further.

As it turned out, answers were below what we regarded as “generally good and correct.” Most people seemed to have left this to the end and perhaps answered under time pressure.
