ECO 317 – Economics of Uncertainty – Fall Term 2009
Week 2 Precept – September 30
EXPECTED UTILITY AND RISK AVERSION – SOLUTIONS

First a recap from the question we considered last week (September 23), namely representing in the probability triangle diagram the version of the Allais paradox we came across in the questionnaire.

In the questionnaire, Question 2 asked you to choose from a pair of lotteries A, B defined by their consequences and probabilities as follows:

A: \$2500 with probability 0.33, \$2400 with probability 0.66, \$0 with probability 0.01 B: \$2400 for sure.

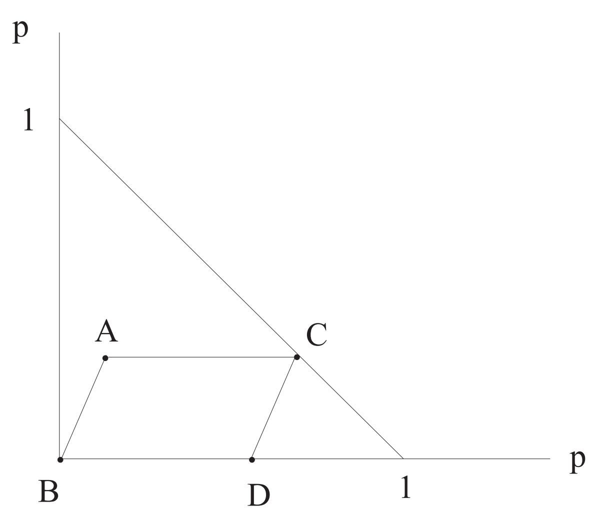
Then Question 3 asked you to choose from a pair of lotteries C, D as follows:

C: \$2500 with probability 0.33, \$0 with probability 0.67 D: \$2400 with probability 0.34, \$0 with probability 0.66

To represent this in the probability triangle diagram, we have three consequences: C_1 is \$0, C_2 is \$2400, and C_3 is \$2500. Remember the diagram has the probabilities p_1 and p_3 (of C_1 and C_3 respectively) as the variables depicted along the X-axis and the Y-axis respectively. Therefore

$$
\begin{array}{lll}
\text{A:} & p_1 = 0.01, & p_3 = 0.33 \\
\text{B:} & p_1 = 0.00, & p_3 = 0.00 \\
\text{C:} & p_1 = 0.67, & p_3 = 0.33 \\
\text{D:} & p_1 = 0.66, & p_3 = 0.00
\end{array}
$$

The points are shown in the figure below. For clarity, the scale has been distorted so that the horizontal difference of 0.01 between A and B or between C and D is made to appear much larger.



The lines AB and CD are parallel. If the independence axiom were valid, indifference curves would be a family of parallel straight lines. Then AB would intersect indifference curves in the same direction as CD. Therefore we should see people preferring A to B if and only if they also preferred C to D. That is, choices should be either A, C or B, D. Choices A, D or B, C are inconsistent with the independence axiom and therefore cannot be explained by expected utility theory.

Of the two possible choice combinations that violate the independence axiom, by far the more common one observed in reality is B, C. For B to be better than A, the indifference curves in this region should be steeper than the line BA; and for C to be better than D, the indifference curves in that region should be flatter than the line CD. That is, higher indifference curves have should have steeper slopes. This pattern of the indifference map is called "fanning out." (Here it does not matter whether the indifference curves are or are not straight lines, merely that they should not be mutually parallel.)

To understand the thinking that seems implicit in the choices that violate expected utility here, consider the changes that transform lottery B into lottery A, and lottery D into lottery C. In each case, two things are being done: [1] 0.33 of probability is being shifted from \$2400 to \$2500, and [2] 0.01 of probability is being shifted from \$2400 to \$0. The difference is that in B the probability of getting nothing is zero. The additional probability 0.01 of losing everything in going to A therefore looms large in peoples minds. By contrast, in D the probability of getting nothing is already quite large, namely 0.66. The additional 0.01 in going to C seems a negligible increment, and in comparison the 0.33 chance of increasing the good payoff by \$100 seems significant. That is why many people dont like A when compared to B, but like C when compared to D, that is, they make the pair of choices B, C.

But expected utility is linear in probabilities; therefore a given increment of probability is valued the same regardless of the initial level of that probability. Thus

$$\begin{aligned} EU(A) - EU(B) &= 0.33\,[u(2500) - u(2400)] - 0.01\,[u(2400) - u(0)] \\ EU(C) - EU(D) &= 0.33\,[u(2500) - u(2400)] - 0.01\,[u(2400) - u(0)] \end{aligned}$$

The two differences are the same, so A should be preferred to B if and only if C is preferred to D. Thus expected utility theory cannot handle a thought process or preference structure that is nonlinear in probabilities, where there may be so to speak diminishing (or increasing) returns to probability.

What is or is not reasonable? We cannot give an answer based on postulates or rationality alone; both are compatible with the basic completeness, transitivity, and continuity axioms. But we will come across some arguments, pro and con, later in the course.

Now we turn to this week's assignment: Questions (1.1)–(1.3) on pp. 24–25 of the Eeckhoudt-Gollier-Schlesinger textbook.

Textbook exercise (1.1):

(a) The initial wealth is 10, and the certainty equivalent e of the lottery is the change that makes the sure prospect $(10 + e)$ indifferent to the expected utility of the lottery. So

$$(10 + e)^{1/2} = \tfrac{1}{2}\,(10 - 6)^{1/2} + \tfrac{1}{2}\,(10 + 6)^{1/2} = \tfrac{1}{2}\,2 + \tfrac{1}{2}\,4 = 3\,.$$

So $e = -1$. The risk premium is the maximum the person would be willing to pay to avoid the risk, therefore the negative of the certainty equivalent, namely 1.

(b) Pratt's formula for the *relative* risk premium (p. 18, eq. (1.15) in the book) is

$$\hat{\Pi}(\tilde{z}) = \tfrac{1}{2}\,\sigma^2\,R(w)$$

where σ^2 is the variance of the proportional risk $\tilde{z}$, and $R(w)$ the coefficient of relative risk aversion. For the utility-of-consequences function $u(w) = w^{1/2}$ we have

$$u'(w) = \tfrac{1}{2}\,w^{-1/2}, \qquad u''(w) = -\frac{1}{4}\,w^{-3/2}, \qquad R(w) = -\frac{-w\,\frac{1}{4}\,w^{-3/2}}{\frac{1}{2}\,w^{-1/2}} = \tfrac{1}{2}\,.$$

And for the risk here, namely that the initial 10 will increase or decrease by the *proportion* 0.6 with equal probability, we have $\sigma = 0.6$. Therefore

$$\hat{\Pi}(\tilde{z}) = \tfrac{1}{2}\,0.36\,\tfrac{1}{2} = 0.09\,.$$

Therefore the *absolute* risk premium equals the relative risk premium multiplied by the initial wealth, that is, $10 * 0.09 = 0.9$. Of course this is an approximation given by Pratt's formula; the exact value as we saw above is 1.

(c) For this function the absolute risk aversion is

$$-\frac{u''(w)}{u'(w)} = -\frac{-\frac{1}{4}\,w^{-3/2}}{\frac{1}{2}\,w^{-1/2}} = \frac{1}{2\,w}$$

which is a decreasing function of w. The relative risk aversion is constant and equal to $\frac{1}{2}$ as we saw above.

(d) With $v(w) = w^{1/4}$, the certainty equivalent of the lottery is given by

$$(10+e)^{1/4} = \tfrac{1}{2}\,(10-6)^{1/4} + \tfrac{1}{2}\,(10+6)^{1/4} = \tfrac{1}{2}\,1.414 + \tfrac{1}{2}\,2 = 1.707$$

So $10 + e = 8.49$ and $e = -1.51$. The risk premium is 1.51.

The new function has constant relative risk aversion equal to $\frac{3}{4} > \frac{1}{2}$, so the risk premium is higher. This relates to the fact that

$$v(w) = [u(w)]^{1/2}\,,$$

or v is an increasing concave transformation of u, so v is "more concave" than u.

(e) Here we go back to the u function, and the risk as a proportion of the initial wealth now has standard deviation only $\sigma = 0.3$ or $\sigma^2 = 0.09$. Therefore Pratt's formula gives

$$\hat{\Pi}(\tilde{z}) = \tfrac{1}{2}\,0.09\,\tfrac{1}{2} = 0.0225\,.$$

The absolute risk premium is (approximately) 0.225. This is only one-fourth of what it was before; the risk premium is proportional to the variance and therefore proportional to the square of the standard deviation.

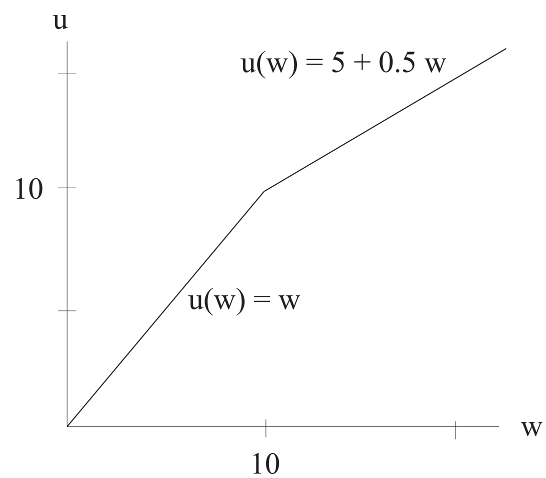
If we use the exact formula,

$$(10+e)^{1/2} = \tfrac{1}{2}\,(10-3)^{1/2} + \tfrac{1}{2}(10+3)^{1/2} = \tfrac{1}{2}\,2.64575 + \tfrac{1}{2}\,3.60555 = 3.12565\,,$$

so $10 + e = 9.76969$ and the exact risk premium is 0.23031.

Textbook exercise (1.2):

(a)The figure below shows the graph of the function. It is globally weakly concave but not strictly concave. The person will be risk-neutral for risk distributions whose support lies entirely below 10 or entirely above 10, but risk-averse when the support spans across 10 (so there are positive probabilities of outcomes below 10 as well as of outcomes above 10).



(b) With initial wealth 10, and a lottery $\tilde{X}$: $(-k, \frac{1}{2}; k, \frac{1}{2})$ for any $k \in (0, 10)$, we have

$$u(10 + e) = \tfrac{1}{2}\,(10 - k) + \tfrac{1}{2}\,[5 + \tfrac{1}{2}\,(10 + k)\,] = 10 - 0.25\,k\,.$$

This is surely < 10, so the correct formula to use for the u function on the left hand side is $u(w) = w$. Therefore $e = -0.25\,k$, and the risk premium is $0.25\,k$.

This can now be used in the question actually asked here, where $k = 6$, and for the one asked in part (d), where $k = 3$. Here the risk premium increases in proportion to the size of the risk, not to the square of the size as Pratt's formula would suggest.

(c) The Arrow-Pratt formulas cannot be used because u is not differentiable at the initial point $w = 10$. Actually, even if the initial point w was different from 10 where the function has its kink (discontinuous first derivative) , the formula would be useless because it would use $u''(w) = 0$ at the bas point and give a zero risk premium: the function is linear in each of its pieces and therefore locally risk-neutral at all points except the one with the kink. Incidentally this answers part (e).

Textbook exercise (1.3):

(a) Here

$$(4 + e)^2 = \tfrac{1}{2}\,(4 - 2)^2 + \tfrac{1}{2}\,(4 + 2)^2 = \tfrac{1}{2}\,4 + \tfrac{1}{2}\,36 = 20\,,$$

so $4 + e = 4.4721$ and $e = 0.4721$. The risk "premium" is -0.4721, negative because the utility-of-consequences function is convex and the decision-maker is a risk-lover.

(b) Now

$$(4 + e)^4 = \tfrac{1}{2}\,(4 - 2)^4 + \tfrac{1}{2}\,(4 + 2)^4 = \tfrac{1}{2}\,16 + \tfrac{1}{2}\,1296 = 656\,,$$

so $4 + e = 5.0609$, $e = 1.0609$, and the risk premium is -1.0609. The risk premium is even more negative than in (a) because the decision-maker is even more risk-loving. This is reflected in the fact that $v(w) = u(w)^2$; the new utility function is more convex than the one in part (a).