ECO 317 – Economics of Uncertainty – Fall Term 2009
Week 3 Precepts – October 7
OPTIMAL CHOICES UNDER RISK – QUESTIONS

Question 1:

On the eve of the football game against Yale, your initial wealth is $W_0 = 1000$. The chance of Princeton winning is 50:50. You can bet at fair odds, taking either side to win, and betting any amount of money you choose. You are an expected utility maximizer, and your utility of consequences depends on your final wealth W and on whether Princeton wins (P) or Yale wins (Y). Specifically,

$$u(W, P) = 2\, W^{1/2}, \qquad u(W, Y) = W^{1/2}\,.$$

(a) Find the optimal amount X that you will bet on Princeton to win (so if X turns out to be negative, you are betting against Princeton).
(b) Having made the optimal bet, do you still prefer Princeton to win?

Question 2:

You are an expected utility maximizer with a logarithmic utility-of-consequences function. Your initial wealth is W_0. You can stake X of this on a bet that will return to you either double the amount you staked or half the amount you staked, with equal probabilities. Find the algebraic expression your optimal choice of X as a function of W_0.

Show that the elasticity of your optimal choice of X with respect to your initial wealth W_0 equals 1. Give an intuitive reason for this result.

Question 3:

Consider a risk-averse expected utility maximizer whose utility-of-consequences function is differentiable. His initial wealth is W_0. He can invest an amount $X \geq 0$ in a risky asset that pays a random total rate of return R with the probability density function $f(R)$ defined over the support $[R_L, R_H]$, and $(W_0 - X)$ in a safe asset that pays a non-random total rate of return R_0. Prove that the optimal X is > 0 if, and only if, $E[R] > R_0$.

Is this behavior "reasonable"? Do you know any empirical observation about choice under risk that contradicts this result? What modification of the stated conditions can reconcile the theory and the empirical finding?

Question 1:

(a) If you bet X for Princeton to win at the fair 50:50 odds, your final wealth will be $W_P = 1000 + X$ and $W_Y = 1000 - X$. Therefore your expected utility is

$$EU = \tfrac{1}{2} * 2 \, (1000 + X)^{1/2} + \tfrac{1}{2} * (1000 - X)^{1/2} \,.$$

To maximize this,

$$\frac{d \, EU}{dx} = \tfrac{1}{2} * (1000 + X)^{-1/2} + \tfrac{1}{2} * \tfrac{1}{2} \, (1000 - X)^{-1/2} \, (-1) = 0 \,.$$

So

$$
\begin{aligned}
\tfrac{1}{2} \, (1000 + X)^{-1/2} &= \tfrac{1}{4} \, (1000 - X)^{-1/2} \\
2 \, (1000 - X)^{1/2} &= (1000 + X)^{1/2} \\
4 \, (1000 - X) &= 1000 + X \\
5 \, X &= 3000 \\
X &= 600
\end{aligned}
$$

(b) With $X = 600$, $W_P = 1600$, $W_Y = 400$, so

$$u(W_P, P) = 2 \, * \, 1600^{1/2} = 2 * 40 = 80, \qquad u(W_Y, Y) = 400^{1/2} = 20$$

So you still prefer Princeton to win.

Question 2:

If you bet X, your final wealth will be $W_0 - X + 2 \, X = W_0 + X$ or $W_0 - X + \tfrac{1}{2} \, X = W_0 - \tfrac{1}{2} \, X$ with equal probabilities. So your expected utility is

$$\tfrac{1}{2} \, \ln(W_0 + X) + \tfrac{1}{2} \, \ln(W_0 - \tfrac{1}{2} X)$$

To maximize this, the first-order condition is

$$\frac{1}{2} \, \frac{1}{W_0 + X} + \frac{1}{2} \, \frac{-1/2}{W_0 - X/2} = 0$$

or

$$W_0 + X = \frac{W_0 - X/2}{1/2}, \qquad W_0 + X = 2 \, (W_0 - X/2) = 2 \, W_0 - X, \qquad 2 \, X = W_0, \qquad X = \tfrac{1}{2} W_0$$

Then the elasticity of X with respect to W_0 is

$$\frac{W_0}{X} \frac{dX}{dW_0} = \frac{W_0}{W_0/2} \frac{1}{2} = 1$$

Intuition: The logarithmic utility-of-consequences function has constant relative risk-aversion; therefore the fraction of your income allocated to the risky asset (the bet) is independent of your initial wealth. Therefore the absolute amount of the bet you choose is proportional to your initial wealth.

Question 3:

The expression for final wealth is

$$W = (W_0 - X) R_0 + X R$$

The investor maximizes expected utility

$$EU(X) = \int_{R_L}^{R_H} u(W) f(R) \, dR$$

Differentiating,

$$EU'(X) = \int_{R_L}^{R_H} u'(W) (R - R_0) f(R) \, dR$$

Because the investor is risk-averse, $U(X)$ is a concave function, and therefore the first-order condition is necessary and sufficient for maximization.

For an optimum at an $X > 0$, the first-order condition is $EU'(X) = 0$. If the optimum is at $X = 0$, the first-order condition is $EU'(0) \leq 0$. Thus $EU'(0) > 0$ is necessary and sufficient for an optimum at $X > 0$. In other words, if $EU'(0) > 0$, then $EU(X)$ starts to increase as X increases starting at 0, so the optimum cannot be at 0. So the answer hinges on the sign of $EU'(0)$.

When $X = 0$, $W = W_0 R_0$ is non-random. Therefore

$$
\begin{aligned}
EU'(0) &= \int_{R_L}^{R_H} u'(W_0 R_0) (R - R_0) f(R) \, dR \\
&= u'(W_0 R_0) \int_{R_L}^{R_H} (R - R_0) f(R) \, dR = u'(W_0 R_0) \{ \, \mathrm{E}[R] - R_0 \, \}
\end{aligned}
$$

Thus $EU'(0) > 0$ if and only if $\mathrm{E}[R] > R_0$. This completes the proof.

This is contradicted by the empirical observations of Thaler, Kahneman-Tversky and others, used by Rabin, that people turn down even very small gambles with expected positive money value. A utility-of-consequences function with a kink at the initial wealth, so that the limit of $u'(W)$ as $W \to W_0$ from the left is greater than the limit of $u'(W)$ as $W \to W_0$ from the right, can reconcile the theory and the empirical finding. We will discuss these ideas in class soon.