ECO 317 – Economics of Uncertainty – Fall Term 2009
Week 5 Precepts – October 21
INSURANCE, PORTFOLIO CHOICE - QUESTIONS

Important Note: To get the best value out of this precept, come with your calculator or computer that can readily calculate the numbers in the various cases below.

Question 1:

You have initial wealth W_0 dollars. With probability p you will suffer a disaster that will wipe out half of it (loss $L = \frac{1}{2} W_0$); otherwise it will stay intact. You can insure against this loss. Denote by q the premium per dollar of insurance. This means that if you buy X dollars of insurance coverage, you have to pay $q\,X$ dollars right now, and will get X dollars from the insurance company if you suffer the disaster and nothing if you do not.

(a) Insurance is supplied by risk-neutral companies in a competitive insurance market. If a claim for X dollars arises, the company must incur an administrative cost of $c\,X$ dollars to investigate and process it. Find the expected profit of an insurance company on a contract for X dollars of insurance coverage. If competition ensures zero expected profit on each such contract, what relation must link q, p, and c?

(b) Suppose you have a utility-of-consequences function with a constant coefficient of relative risk aversion r. Find the expression for your expected utility when you buy X dollars of insurance coverage.

(c) By maximizing this expected utility with respect to X, find a formula for the fraction $X/(0.5\,W_0)$ of your loss that you will choose to cover, as a function of q, p, and r.

(d) Numerically evaluate this, taking $p = 0.1$, two cases of c, namely $c = 0.1$ and $c = 0.2$, and three cases of r, namely $r = 0.25$, $r = 1$, and $r = 10$ (six calculations in all). In each case, the price of insurance q is to be set at its competitive equilibrium level. Explain your finding for the case $r = 0.25$, $c = 0.2$.

Question 2:

Note: Do all the calculations for this problem in units of $1 million (megabucks). Be especially careful with your algebra and arithmetic in this question.

Your initial wealth is $1 million. You can invest a proportion x of it in stocks, a proportion y in bonds, and the rest in cash. There are five "states of the world" or "scenarios." Cash always yields zero return, therefore that part of your wealth stays at $(1 - x - y)$ in all scenarios. The scenarios are as follows:

(1) The Ho-Hum Scenario (probability 40%): The values of stocks and bonds do not change at all.

(2) The Goldilocks Economy (probability 20%): Everything is exactly right; the economy prospers and inflation is low. The value of stocks doubles, and that of bonds goes up by 50%.

(3) Stagflation (probability 20%): The economy stagnates and interest rates go up. The value of stocks halves, and that of bonds goes down by 25%.

(4) Inflation (probability 10%): The economy booms but interest rates rise sharply. The value of stocks doubles, but that of bonds goes down by 25%.

(5) Deflation (probability 10%): The economy does badly and interest rates are low. The value of stocks halves, but that of bonds goes up by 50%.

Ignore the dividends on the stocks and the interest on the bonds; these are negligible compared to the changes in the values of the assets stated above.

(a) Write down expressions for your final wealth, denoted by respectively W_1, W_2, ... W_5, in each of these five scenarios, in each case as a function of x and y.

(b) Suppose your utility-of-consequences function is

$$U(W) = W - \tfrac{1}{4}\, W^2\,.$$

Write down the expression for your expected utility, as a function of W_1, W_2, ... W_5.

(c) Find the values of x and y that maximize your expected utility. (Do not worry about second-order conditions or boundary solutions in this part.)

(d) Do not derive any calculus second-order conditions, but say in a couple of sentences why expected utility is here a concave function of (x, y) ensuring that the SOCs are satisfied.

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Question 1:

(a) The expected profit of the insurance company is $q\,X - p\,X - p\,c\,X = [q - (1+c)p]\,X$. (You might have thought that the administration cost was cX. But remember, this cost is incurred only if a claim is made, that is, with probability p. Therefore the expected cost is pcX.) With several perfectly competitive insurance companies, in equilibrium the expected profit must be zero, therefore $q = (1+c)\,p$.

(b) Call the scenario in which you suffer the loss scenario 1, and the one where your wealth stays intact, scenario 2. If you take out coverage X, your expected utility is

$$EU = \frac{1}{1-r}\;\left[p\,\{\tfrac{1}{2}\,W_0 + (1-q)X\}^{1-r} + (1-p)\,\{W_0 - qX\}^{1-r}\right]$$

when $r \neq 1$ (and the log case if $r = 1$).

(c) To choose X to maximize this, the FONC is

$$\frac{dEU}{dX} = p\,\{\tfrac{1}{2}\,W_0 + (1-q)X\}^{-r}\,(1-q) + (1-p)\,\{W_0 - qX\}^{-r}\,(-q) = 0$$

(The derivative dEU/dX has the same functional form for all cases of r, so you actually don't need to do the log case separately.)

Then

$$\frac{\tfrac{1}{2}\,W_0 + (1-q)\,X}{W_0 - q\,X} = \left(\frac{(1-p)\,q}{p\,(1-q)}\right)^{-1/r}$$

Write z for the right hand side. Then the expression for the fraction of your loss that is covered:

$$\frac{X}{0.5\,W_0} = \frac{2\,z - 1}{1 - q + q\,z}$$

Second-order conditions are OK because the wealth in each scenario is a linear function of X, the utility-of-consequences function in each scenario is a concave function of the wealth in that scneario, and expected utility is a positive linear combination of the utilities in the various scenarios.

(d) When $p = 0.10$, for two values of the administrative cost factor $c = 0.1$ and 0.2, we have $q = 0.11$ and 0.12 respectively. Then, for the three values of the risk aversion coefficient r given, we have the following table for the resulting values of the coverage ratio $X/(\frac{1}{2}\,W_0)$:

	Relative risk aversion r		
Cost factor c	0.25	1.0	10.0
0.1	0.306	0.798	0.979
0.2	– 0.126	0.643	0.962

The formula yields negative coverage for the case $r = 0.25$, $c = .2$ because with such low risk aversion and the high load factor, you would not wish to purchase any insurance. The true optimum is an extreme of $X = 0$.

Question 2:

(a) Expressions for the final wealth in the five scenarios

$$\begin{aligned}
W_1 &= (1 - x - y) + x + y = 1 \\
W_2 &= (1 - x - y) + 2\ x + 1.5\ y = 1 + x + 0.5\ y \\
W_3 &= (1 - x - y) + 0.5\ x + 0.75\ y = 1 - 0.5\ x - 0.25\ y \\
W_4 &= (1 - x - y) + 2\ x + 0.75\ y = 1 + x - 0.25\ y \\
W_5 &= (1 - x - y) + 0.5\ x + 1.5\ y = 1 - 0.5\ x + 0.5\ y
\end{aligned}$$

(b) Expected utility

$$\begin{aligned}
EU &= 0.4\ [W_1 - \tfrac{1}{4}\ (W_1)^2] + 0.2\ [W_2 - \tfrac{1}{4}\ (W_2)^2] + 0.2\ [W_3 - \tfrac{1}{4}\ (W_3)^2] \\
&\quad + 0.1\ [W_4 - \tfrac{1}{4}\ (W_4)^2] + 0.1\ [W_5 - \tfrac{1}{4}\ (W_5)^2]
\end{aligned}$$

(c)

NOTE: I keep the expression for EU as in (b) and use the chain rule to differentiate with respect to x and y. This is easier, and less liable to error, than expanding out EU explicitly in terms of x and y and then differentiating.

The FONCS for EU-maximization:

$$\begin{aligned}
\frac{\partial EU}{\partial x} &= 0.2\ [1 - \tfrac{1}{2}\, W_2]\ \frac{\partial W_2}{\partial x} + 0.2\ [1 - \tfrac{1}{2}\, W_3]\ \frac{\partial W_3}{\partial x} \\
&\quad + 0.1\ [1 - \tfrac{1}{2}\, W_4]\ \frac{\partial W_4}{\partial x} + 0.1\ [1 - \tfrac{1}{2}\, W_5]\ \frac{\partial W_5}{\partial x} \\
&= 0.2\ [1 - \tfrac{1}{2}\,(1 + x + 0.5\ y)]\ (1) + 0.2\ [1 - \tfrac{1}{2}\,(1 - 0.5\ x - 0.25\ y)]\ (-0.5) \\
&\quad + 0.1\ [1 - \tfrac{1}{2}\,(1 + x - 0.25\ y)]\ (1) + 0.1\ [1 - \tfrac{1}{2}\,(1 - 0.5\ x + 0.5\ y)]\ (-0.5) \\
&= 0.075 - 0.1875\ x - 0.0375\ y = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial EU}{\partial y} &= 0.2\ [1 - \tfrac{1}{2}\, W_2]\ \frac{\partial W_2}{\partial y} + 0.2\ [1 - \tfrac{1}{2}\, W_3]\ \frac{\partial W_3}{\partial y} \\
&\quad + 0.1\ [1 - \tfrac{1}{2}\, W_4]\ \frac{\partial W_4}{\partial y} + 0.1\ [1 - \tfrac{1}{2}\, W_5]\ \frac{\partial W_5}{\partial y} \\
&= 0.2\ [1 - \tfrac{1}{2}\,(1 + x + 0.5\ y)]\ (0.5) + 0.2\ [1 - \tfrac{1}{2}\,(1 - 0.5\ x - 0.25\ y)]\ (-0.25) \\
&\quad + 0.1\ [1 - \tfrac{1}{2}\,(1 + x - 0.25\ y)]\ (-0.25) + 0.1\ [1 - \tfrac{1}{2}\,(1 - 0.5\ x + 0.5\ y)]\ (0.5) \\
&= 0.0375 - 0.0375\ x - 0.046875\ y = 0
\end{aligned}$$

You can solve these as they are; they also simplify to

$$5\ x + y = 2, \qquad 4\ x + 5\ y = 4$$

The solutions are

$$x = 2/7 = 0.286, \qquad y = 4/7 = 0.571$$

(Then the fraction held in cash is $1 - x - y = 1/7 = 0.143$.)

(d) The wealth in each scenario is a linear function of x and y, the utility-of-consequences in each scenario is a concave function of the wealth in that scenario, and expected utility is a positive linear combination of the state-by-state utilities. Therefore expected utility is a concave function of (x, y). (This argument is similar to the ones you used in ECO 310 to establish curvature properties of various "indirect" functions, such as a firm's cost function or a consumer's indirect utility function.)

Note that in some scenarios the final wealth is decreasing in x and/or y. But what matters for second-order condition is concavity; that is a property of the second derivatives. All linear terms in x and y are going to disappear, and for the quadratic terms the increasing/decreasing distinction is not going to matter.