ECO 317 – Economics of Uncertainty – Fall Term 2009
Week 10 Precepts – December 2

Algebra of the Rothschild-Stiglitz Model – Questions

In class we treated the Rothschild-Stiglitz model of adverse selection in insurance using geometric methods (Lecture Note 15). Here we develop a parallel algebraic treatment of the basic model, and then extend it to study a variant that takes into account an important feature of the legal reality of insurance contracts.

Use the following setup, notation, and assumptions: There are two states of the world. State 1 is the no-loss state, and state 2 is the loss state. Each individual has wealth W_0 and would lose D of it in state 2, so his endowments of wealth (Arrow-Debreu securities) in the two states are W_0 and $W_0 - D$ respectively. All individuals are risk-averse and have identical utility-of-consequences functions $u(W)$. There are two risk types of individuals, H and L, with probabilities of loss $1 > \pi_H > \pi_L > 0$. Insurance companies are risk-neutral, perfectly competitive and have zero administrative costs. Each firm can offer only one contract. There is free entry of firms.

The basic model

Begin by observing that there cannot be pooling of types in equilibrium. This is p. 6 of Lecture Note 15. Take this as given, and focus on equilibria where there are two contracts, one intended for being taken up by each type. Proceed as follows:

(a) Show that each contract must make zero expected profit.

(b) Show that the contract intended for type H will have full insurance at the actuarially fair premium. This is a verbal argument, no math needed. After stating the verbal argument, write down expressions for the premium and the indemnity that H-type individuals get in this contract, and their resulting utility. Use the following notation. The premium is P_H, and the indemnity is $(P_H + X_H)$, so in effect the insured individual pays the company P_H in state 1 and the company pays the insured X_H in state 2.

(c) Show that the contract intended for type L will maximize the expected utility of this type, subject to the constraints that (i) the insurance company does not make an expected loss, and (ii) the H-type does not prefer to take up the contract intended for the L-type. This is also a verbal argument, no math needed.

(d) Now set up the constrained maximization problem in (b). Use the following notation. The premium is P_L, and the indemnity is $(P_L + X_L)$, so in effect the insured individual pays the company P_L in state 1 and the company pays the insured X_L in state 2. Just set up the problem; no need to write down the Kuhn-Tucker conditions etc.

The extension

The basic model assumes that an applicant for insurance has no obligation to disclose any private information about his risk; it is the insurance company's job to design a screening mechanism as best as it can. In reality, most legal systems do expect the insured to make a full disclosure in of any information he has pertaining to his riskiness. This is the legal doctrine of *uberrima fides*, or utmost good faith. If he fails to make such a disclosure, and the insurance company finds out later that the customer had withheld information, the company can abrogate the contract ex post facto, and refuse to pay the covered sum (indemnity). Therefore the company need not check any statements made by the client in the insurance application; it need carry out the check only if a claim is filed. This reduces the cost of checking by the factor of the probability of loss, and therefore be advantageous. Here you are asked to deduce the consequences of such a law for the equilibrium.

The most general contract that the insurance company can write, consistent with the law and the information limitation, takes the following form: The customer must declare on his application his risk type $t = H$ or L. If type t has been declared, the customer pays the company P_t. Then, if there is an accident, with probability a_t company will investigate the customer's assertion. The cost of each investigation is c. If the company investigates and finds that the customer has been truthful, it pays the customer $P_t + X_t$. If if investigates and finds that the customer has been untruthful, it pays the customer $P_t + Y_t$. If it does not investigate at all, it pays the customer $P_t + Z_t$. Thus the company has 10 choice variables in specifying a contract: P_t, a_t, X_t, Y_t, and Z_t for $t = L$ and H.

An insurance company wants to maximize its expected profit. But competition ensures that in equilibrium each will make zero expected profit.

You can think of a pooling contract as one that does not require a declaration of type, but the analysis is kept simpler and more streamlined by thinking of a pooling contract as a special case of the above specification, with $P_L = P_H$, $a_L = a_H$ etc.

(a) First show that in equilibrium there cannot be a pooling of types. This is again a purely verbal argument. Then it follows as in the first part that in any separating equilibrium, each contract must make zero expected profit.

(b) Show that the contract intended for type H will offer full insurance at a statistically fair premium, and no investigation will be carried out in the event of an accident. That is, $P_H = \pi_H D$, $a_H = 0$, $Z_H = D$,and X_H, Y_H are irrelevant. This can again be a purely verbal argument

(c) Show that the contract intended for type L maximizes this type's expected utility subject to the constraints that (i) the firm does not make an expected loss, and (ii) the H-type does not prefer to claim to be L-type. This is again a verbal argument

(d) Set up the constrained maximization problem mathematically. Show that this means choosing P_L, a_L, X_L, Y_L, and Z_L to solve the following problem:

$$\text{Maximize} \;\; (1 - \pi_L)\, u(W_0 - P_L) + \pi_L \; [\; a_L \, u(W_0 - D + X_L) + (1 - a_L)\, u(W_0 - D + Z_L) \;]$$

subject to the no-loss constraint

$$(1 - \pi_L)\, P_L - \pi_L \; [\; a_L\, (c + X_L) + (1 - a_L)\, Z_L \;] \geq 0$$

and the H-type's incentive compatibility (truthtelling) constraint

$$u(W_0 - \pi_H\, D) \geq (1 - \pi_H) \; u(W_0 - P_L) + \pi_H \; [\; a_L \; u(W_0 - D + Y_L) + (1 - a_L) \; u(W_0 - D + Z_L) \;]$$

(Note that you are given the answer, but have to develop the steps in the argument and interpret the various terms to show why this is the answer.) Verify that the incentive-compatibility condition for type L is automatically satisfied by the solution to this problem.

(e) Construct the Lagrangian expression for this constrained maximization, writing μ_L for the Lagrange multiplier on the no-loss constraint and λ_H for the Lagrange multiplier on the incentive-compatibility constraint of the H-type. Write down the partial derivatives of the Lagrangian with respect to the five choice variables P_L, a_L, X_L, Y_L, and Z_L.

(f) Show that

(1) it is optimal to keep Y_L as low as the law will allow. (From here on, assume this lower bound to be zero, and assume that the maximization with respect to P_L, X_L and Z_L has regular interior solutions.)

(2) the incentive-compatibility constraint of the H type must be binding. (Hint: If it is slack, then by complementary slackness $\lambda_H = 0$. Show that this leads to a contradiction.)

(3) $a_L < 1$

(4) the first-order conditions of the maximization problem in (a) imply

$$\begin{aligned} u'(W_0 - P_L) &= \frac{\mu_L}{1 - \lambda_H \; (1 - \pi_H)/(1 - \pi_L)} \\ u'(W_0 - D + X_L) &= \mu_L \\ u'(W_0 - D + Z_L) &= \frac{\mu_L}{1 - \lambda_H \; \pi_H / \pi_L} \end{aligned}$$

Hence show that

$$W_0 - D + X_L > W_0 - P_L > W_0 - D + Z_L$$

(g) Suggest when $a_L = 0$ is possible.

(h) Give an intuition for the inequalities in (f)(4) satisfied by the equilibrium contract for the L-types.

(i) Is this equilibrium better in some sense than the equilibrium in the original Rothschild-Stiglitz model? If so, how?

(j) Is this a good model of the legal requirement of full disclosure in good faith? Suggest some ways to improve it.

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ALGEBRA OF THE ROTHSCHILD-STIGLITZ MODEL – SOLUTIONS

The basic model

(a) In equilibrium, a contract that yields negative expected profit cannot last because the firm offering it will exit. A contract yielding positive expected profit cannot last either, because a risk-neutral new entrant firm can offer a slightly better deal to the same customers while still making positive expected profit. Note that the assumption that each firm can offer only one contract is important here. If a firm could offer multiple contracts, then in some circumstances a firm could offer a pair of contracts that jointly make zero expected profits but involve some cross-subsidy, to beat out a pair of existing separately zero-expected-profit contracts. This further reduces the possibility of a separating equilibrium. We will leave this issue out for more advanced courses.

(b) If this was not the case, then in the space of Arrow-Debreu state-contingent wealths (W_1, W_2), the company's no-loss region and the H-type's no-worse region must have a non-empty intersection. Then an entrant could offer a slightly better contract and make a positive expected profit. If any L-types take up this contract, that only increases the expected profit (although in fact they won't).

Full insurance means $W_0 - P_H = W_0 - D + X_H$, so $X_H = D - P_H$. And zero expected profit means $P_H = \pi_H (P_H + X_H) = \pi_H D$. So the individual ends up with $(W_0 - \pi_H D)$ in either state, and his utility is $u(W_0 - \pi_H D)$

(c) If this was not the case, in the (W_1, W_2) space the company's no-loss region, the H-type's truthful selection region, and the L-type's no-worse region would have a non-empty intersection. So an entrant could offer a contract that (i) the L-types would prefer to the one in the supposed equilibrium, (ii) the H-types would not prefer to their fair full insurance, and (iii) make the company positive expected profit.

(d) The problem is to choose P_L, X_L to maximize

$$(1 - \pi_L)\; u(W_0 - P_L) + \pi_L \; u(W_0 - D + X_L)$$

subject to

$$(1 - \pi_L)\, P_L + \pi_L \; (-X_L) \geq 0\,,$$

and

$$u(W_0 - \pi_H D) \geq (1 - \pi_H)\; u(W_0 - P_L) + \pi_H \; u(W_0 - D + X_L)\,.$$

The Kuhn-Tucker conditions for this problem are somewhat uninteresting because there are only two choice variables and two inequality constraint. In the interesting case both constraints will be binding, so the solution is given by solving the two equality constraints regarding the contract parameters P_L and X_L as two unknowns.

The extension

(a) If there is a pooling contract, a costly ex post investigation is pointless and therefore such a contract must have $a_L = a_H = 0$. But then such a contract is no different than the pooling contract in the standard Rothschild-Stiglitz model and can be beaten out by their separation contract. The H-types in that model do not choose this, even though it has no risk of being detected to be withholding information. So the company proposing such a deviating contract can still set $a_L = a_H = 0$ and attract only the low-risk types. (Note – here it is important to pay attention to this additional issue of specification of the a's in the contracts.)

(b) The argument is the same as that in part (a) of the basic model. The only additional point to be noted is that there is no need to specify any costly investigation when the type taking up the contract is the H-type.

(c) The argument is the same as that for part (b) of the basic model. The only difference is that there are more parameters of the contract to choose, so the Kuhn-Tucker conditions are going to be of interest.

(d) The company gets P_L from a customer who declares type L. When the incentive compatibility constraint for the H-types is also present, separation is achieved, no H-type chooses this contract. With probability π_L the accident occurs. Subject to this happening, with probability a_L the company carries out an investigation, and pays out a total of $c + (P_L + X_L)$ (remember that no lying H-types are actually there to be caught). With probability $(1 - a_L)$ the company makes no investigation and pays out $(P_L + Z_L)$. Thus the non-negative expected profit constraint is

$$P_L - \pi_L \; [\; a_L \; (c + P_L + X_L) + (1 - a_L) \; (P_L + Z_L) \;] \geq 0$$

or

$$(1 - \pi_L) \; P_L - \pi_L \; [\; a_L \; (c + X_L) + (1 - a_L) \; Z_L \;] \geq 0 \tag{1}$$

The expected utility of type L is found by constructing his final wealth in the various eventualities: With probability $(1 - \pi_L)$ he has no accident and ends up with $W_0 - P_L$. With probability π_L he has an accident. Conditional on this, with probability a_L there is an investigation which reveals him to be telling the truth, so he ends up with $W_0 - P_L - D + (P_L + X_L) = W_0 - D + X_L$, whereas with probability $(1 - a_L)$ there is no investigation, and he ends up with $W_0 - P_L - D + (P_L + Z_L) = W_0 - D + Z_L$. So his expected utility is

$$(1 - \pi_L) \; u(W_0 - P_L) + \pi_L \; [\; a_L \, u(W_0 - D + X_L) + (1 - a_L) \, u(W_0 - D + Z_L) \;]$$

An H-type who declares the type truthfully gets full insurance at a statistically fair price – premium $P_H = \pi_H \, D$, indemnity D, and no investigation. So he ends up with $W_0 - \pi_H \, D$ in either state, and gets utility $u(W_N - \pi_H \, D)$.

If the H-type declares himself to be L type, then with probability $(1 - \pi_H)$ he does not have an accident and ends up with $W_0 - P_L$. With probability π_H he has an accident; then with probability a_L there is an investigation in which he is found out and ends up with $W_0 - P_L - D + (P_L + Y_L) = W_0 - D + Y_L$, whereas with probability $(1 - a_L)$ there is no investigation and he ends up with $W_0 - P_L - D + (P_L + Z_L) = W_0 - D + Z_L$. So type-$H$'s incentive compatibility constraint is

$$u(W_0 - \pi_H \, D) \geq (1 - \pi_H) \; u(W_0 - P_L) + \pi_H \; [\; a_L \; u(W_0 - D + Y_L) + (1 - a_L) \; u(W_0 - D + Z_L) \;] \tag{2}$$

Important note – even if type H declares himself to be type L, his accident probability remains π_H and he knows it, so this is the probability we must use on the right hand side.

Under perfect competition, the contract intended for the L-type should maximize their expected utility subject to the no-expected-loss constraint and the two types' incentive compatibility constraints. Suppose we first solve the problem by ignoring the incentive compatibility constraint for type L. The standard Rothschild-Stiglitz separating contract for the low risk types is a special case of this, with a_L constrained to be zero. Therefore the maximized expected utility of the L types in our problem can be no less than that in the Rothschild-Stiglitz model. But that was already known to be higher than the utility this type would get from pretending to be H-type – the L-type's incentive compatibility constraint was not binding in the Rothschild-Stiglitz model. Therefore *a fortiori* that constraint is not binding in our model.

This establishes that the constrained optimization problem is as stated in the question.

(e) The Lagrangian is

$$\begin{aligned}
\mathcal{L} \;\; = \;\; & (1 - \pi_L) \; u(W_0 - P_L) + \pi_L \; [\; a_L \, u(W_0 - D + X_L) + (1 - a_L) \, u(W_0 - D + Z_L) \;] \\
& + \mu_L \; \{ \; (1 - \pi_L) \; P_L - \pi_L \; [\; a_L \; (c + X_L) + (1 - a_L) \; Z_L \;] \; \} \\
& + \lambda_H \; \{ \; u(W_0 - \pi_H \, D) - (1 - \pi_H) \; u(W_0 - P_L) - \pi_H \; [\; a_L \; u(W_0 - D + Y_L) + (1 - a_L) \; u(W_0 - D + Z_L) \;] \; \}
\end{aligned}$$

The partial derivatives of the Lagrangian are:

$$\partial\mathcal{L}/\partial P_L \;\; = \;\; -(1 - \pi_L) \, U'(W_0 - P_L) + (1 - \pi_L) \, \mu_L + (1 - \pi_H) \, \lambda_H \, U'(W_0 - P_L) \tag{3}$$

$$\begin{aligned}
\partial\mathcal{L}/\partial a_L \;\; = \;\; & \pi_L \, [\, U(W_0 - D + X_L) - U(W_0 - D + Z_L) \,] - \mu_L \, \pi_L \, (c + X_L - Z_L) \\
& + \lambda_H \, \pi_H \, [\, U(W_0 - D + Z_L) - U(W_0 - D + Y_L) \,]
\end{aligned} \tag{4}$$

$$\partial\mathcal{L}/\partial X_L \;\; = \;\; \pi_L \; a_L \; [\, U'(W_0 - D + X_L) - \mu_L \,] \tag{5}$$

$$\partial\mathcal{L}/\partial Y_L \;\; = \;\; -\lambda_H \, \pi_H \, a_L \, U'(W_0 - D + Y_L) \tag{6}$$

$$\partial\mathcal{L}/\partial Z_L \;\; = \;\; (1 - a_L) \, [\, \pi_L \, U'(W_0 - D + Z_L) - \pi_L \, \mu_L - \lambda_H \, \pi_H \, U'(W_0 - D + Z_L) \,] \tag{7}$$

(f) (1) From (6) we see that $\partial\mathcal{L}/\partial Y_L$ is always < 0, so it is optimal to keep Y_L as low as the law will allow. (The *uberrima fides* requirement is enforced with varying degrees of strictness in different countries. The U.S. is the least stringent, the U.K. more so, and France is even stricter – if you are found to have intentionally lied, you can be fined, in addition to losing coverage, so there Y_L can be negative.)

(f) (2) If the incentive compatibility constraint for the H type (equation (2)) is slack, then by complementary slackness the multiplier λ_H is zero. Then the first-order conditions for P_L, X_L and Z_L obtained by setting the corresponding derivatives of $\mathcal{L}$ equal to zero imply

$$u'(W_0 - P_L) = u'(W_0 - D + X_L) = u'(W_0 - D + Z_L) = \mu_L$$

Therefore

$$W_0 - P_L = W_0 - D + X_L = W_0 - D + Z_L \, ,$$

and the L-types would have full insurance, which under competition must be actuarially fair. Also, (4) would then imply

$$\partial\mathcal{L}/\partial a_L = -\mu_L \, \pi_L \, c < 0 \, ,$$

so $a_L = 0$. Such a contract would offer full insurance at the low-risk odds without investigation; but then the contract would appeal to the H-types in the standard Rothschild-Stiglitz manner, so their incentive constraint would be violated.

(f) (3) If $a_L = 1$, then an H-type claiming to be an L-type, if he makes a claim, will be investigated for sure, and found to be lying. Therefore he will end up with $W_0 - P_L$ if there is no accident, and because $Y_L = 0$, he will end up with $W_0 - D$ if there is an accident. He would do better buying no insurance at all, and therefore better still by taking the full, fair contract C_H^* intended for his type. But that would mean that the incentive constraint of the H-type would not be binding. That would imply $\lambda_H = 0$, which would lead to a contradiction as proved above in (f) (2).

(f) (4) Simple algebraic manipulation of the first order conditions (3), (5), and (7) yields the stated form:

$$\begin{aligned} u'(W_0 - P_L) &= \frac{\mu_L}{1 - \lambda_H \; (1 - \pi_H)/(1 - \pi_L)} \\ u'(W_0 - D + X_L) &= \mu_L \\ u'(W_0 - D + Z_L) &= \frac{\mu_L}{1 - \lambda_H \; \pi_H/\pi_L} \end{aligned}$$

Since $\pi_H > \pi_L$, we have

$$\pi_H/\pi_L > 1 > (1 - \pi_H)/(1 - \pi_L),$$

and therefore

$$U'(W_0 - D + X_L) < U'(W_0 - P_L) < U'(W_0 - D + Z_L)$$

or

$$W_0 - D + X_L > W_0 - P_L > W_0 - D + Z_L \tag{8}$$

(g) If $a_L = 0$, then this is just the standard Rothschild-Stiglitz problem with no investigation, and the Z_L is just the limited coverage offered to the L-type there. With the new freedom of extra choice variables available, this can be optimal only if the cost of investigation c is sufficiently large.

(h) Some properties of this contract: [1] The inequalities in (8) together with $Y_L = 0$ say that the customer who declares himself to be type L is best off in the event an investigation is carried out and he is found to be telling the truth, the next best off in the outcome where there is no accident, then the outcome where there is an accident and the company does not investigate the truthtelling, and worst off in the event there is an accident and an investigation of the declaration which reveals that the customer had lied. The first and the last are new instruments that encourage truthtelling and deter lying. The latter is made possible by the provision of law that the onus of good faith declaration is on the customer. The former – the reward for truthtelling – is a voluntary optimal choice for the company that comes out of the math, and having obtained the result, one can see that it makes intuitive sense – it helps achieve separation. But I have not seen it used in actual insurance contracts. [2] Investigation of truthtelling is probabilistic ($a_L < 1$). This is a realistic feature of most such investigations, e.g. tax audits.

(i) [1] The argument in the last para of part (d) showed that the expected utility of the L-types under this contract is generally higher than that under the standard Rothschild-Stiglitz contract without investigation, basically because it has more instruments to do the constrained optimization. Thus the L-types do better with the legal requirement of the onus of good faith declaration. The H types get the same full, fair insurance at their probability and without any investigation under both systems. Thus the legal requirement achieves a Pareto improvement. [2] It can also be shown that the separating equilibrium exists for more values of the parameters with this legal requirement than would without it (in the standard Rothschild-Stiglitz model).

(j) Some defects of this model: [1] It is assumed here that a customer knows his type with certainty. This may not be the case. Then a distinction must be made between a customer who claims to be type L with genuinely held belief, and one who knows himself to be type H but knowingly misrepresents type. The law generally treats the former much more mildly than the latter. [2] A basic problem with probabilistic audits is that once they have achieved their purpose of deterring the bad types, there is no actual ex post incentive for the company to spend resources to carry them out. Of course if the bad types know and expect this ahead of time, it won't deter them in the first place. So the company has to have a way to commit itself to carrying out investigations with the stated probability. In reality this may or may not be possible depending on the exact circumstances.