

ECO 317 – Economics of Uncertainty – Fall Term 2009
Problem Set 4 – Answer Key

The distribution was as follows:

100	90-99	< 00
5	9	1

A great job; no significant common errors to point out. If you have really learned Arrow-Debreu theory this well, you have taken a good first step on the path of your seven-figure finance career.

Question 1: 30 points

(a) (3 points) Sempronius' holding of contingent claims goes from (10,10) to (13,8); his expected utility goes

$$\begin{array}{ll} \text{from} & 0.5 \ln(10) + 0.5 \ln(10) = 2.3026 \\ \text{to} & 0.5 \ln(13) + 0.5 \ln(8) = 2.3222 \end{array}$$

(3 points) Jacobus' holding of contingent claims goes from (15,5) to (12,7); his expected utility goes

$$\begin{array}{ll} \text{from} & 0.5 \ln(15) + 0.5 \ln(5) = 2.1587 \\ \text{to} & 0.5 \ln(12) + 0.5 \ln(7) = 2.2154 \end{array}$$

So both individuals gain from the trade.

(b) (6 points) For each individual, the expression for the marginal rate of substitution is

$$-\left. \frac{dC_2}{dC_1} \right|_{EU \text{ constant}} = \frac{0.5 (1/C_1)}{0.5 (1/C_2)} = \frac{C_2}{C_1}$$

The numerical values are $8/13 = 0.6154$ for Sempronius, and $7/12 = 0.5833$ for Jacobus. Therefore the MRS are not quite equal, and the final consumption allocation of contingent consumption is not quite Pareto efficient. If contingent claims in fractional ducats can be exchanged, both can do even better.

(c) (12 points) Equating the MRS values means equating C_2/C_1 for the two. Call the common value k . Using (S) as the label for Sempronius and (J) for Jacobus, we have

$$C_2(S) = k C_1(S), \quad C_2(J) = k C_1(J)$$

The total supply of the consumption good (ducats) is $10 + 15 = 25$ in state 1 and $10 + 5 = 15$ in state 2. Therefore for feasibility of the allocations,

$$15 = C_2(S) + C_2(J) = k [C_1(S) + C_1(J)] = 25 k .$$

So $k = 15/25 = 0.6$. Therefore the set of Pareto-efficient allocations can be described as followed:

For $x \in (0, 15)$, Sempronius gets contingent consumption amounts x in state 1 and $0.6x$ in state 2; Jacobus gets $(25 - x)$ in state 1 and $15 - 0.6x$ in state 2.

(d) (6 points) If the probabilities change to 0.4 for state 1 and 0.6 for state 2, the MRS expression for each individual changes to

$$-\left. \frac{dC_2}{dC_1} \right|_{EU \text{ constant}} = \frac{0.4 (1/C_1)}{0.6 (1/C_2)} = \frac{2}{3} \frac{C_2}{C_1}$$

But for the MRS to be equal across individuals, we still need C_2/C_1 to be equal, so the derivation of the Pareto efficient set is unchanged.

Question 2: (35 points)

(a) (2 points) Scenario 1: Dotcom prospers; total wealth = 60 (megabucks)

Scenario 2: Dotcom is toast; total wealth = 10

(b) (4 points) Endowments: Doc - (10,10); Geek (50,0).

Values Doc: $10 P_1 + 10 P_2$; Geek: $50 P_1$

(c) (4 points) Budget constraints (which will hold as equalities):

$$\begin{aligned} \text{Doc:} \quad & P_1 W_1^D + P_2 W_2^D \leq 10 P_1 + 10 P_2 \\ \text{Geek:} \quad & P_1 W_1^G + P_2 W_2^G \leq 50 P_1 \end{aligned}$$

(d) (4 points) Expected utilities:

$$\begin{aligned} \text{Doc:} \quad & 0.4 \ln(W_1^D) + 0.6 \ln(W_2^D) \\ \text{Geek:} \quad & 0.4 \ln(W_1^G) + 0.6 \ln(W_2^G) \end{aligned}$$

(e) (12 points) Demand functions (use standard Cobb-Douglas method)

$$\begin{aligned} \text{Doc:} \quad & W_1^D = 0.4 \frac{10 P_1 + 10 P_2}{P_1} = 4 + 4 \frac{P_2}{P_1} \\ & W_2^D = 0.6 \frac{10 P_1 + 10 P_2}{P_2} = 6 \frac{P_1}{P_2} + 6 \\ \text{Geek:} \quad & W_1^G = 0.4 \frac{50 P_1}{P_1} = 20, \quad W_2^G = 0.6 \frac{50 P_1}{P_2} = 30 \frac{P_1}{P_2} \end{aligned}$$

(e) (5 points) For equilibrium in the market for scenario-1 Arrow-Debreu securities:

$$4 + 4 \frac{P_2}{P_1} + 20 = 60, \quad \text{so} \quad \frac{P_2}{P_1} = 9$$

(We don't need to look at equilibrium in the other market, and don't need the absolute prices of the two securities, for the usual reasons – Walras' Law and homogeneity.)

(4 points) Then the demand functions give the final wealth amounts:

$$\begin{array}{ll} \text{Doc:} & W_1^D = 40, \quad W_2^D = 6\frac{2}{3} \\ \text{Geek:} & W_1^G = 20, \quad W_2^G = 3\frac{1}{3} \end{array}$$

Additional information: Observe that Geek has twice as much expected wealth as Doc: $0.4 \times 50 + 0.6 \times 0 = 20$ versus $0.4 \times 10 + 0.6 \times 10 = 10$. But Doc ends up with twice as much as Geek in each scenario, because he has more wealth in scenario 2, which is much more valuable.

Comment on the price-taking assumption: When there are only two people trading with each other, you expect them to be aware of this and engage in bargaining, not passive price-taking. So something like a core, or a more refined theory of bargaining leading to a unique outcome, would be better than the supply-demand-equilibrium model. You could suppose that each of these two stands for hundreds of similar people with identical demands. That would work fine in the exchange problems, for example Red and Blue exchanging goods X and Y in Slides 11, where we can think of hundreds of identical Reds meeting hundreds of identical Blues. But here an additional issue arises. For there to be just two scenarios even when there are hundreds of Geeks, the nature of uncertainty must be such that all the dotcoms either succeed or fail together. That is, the uncertainty about the success of all the Geeks' dotcomes must be perfectly correlated. Otherwise, with n Geeks, one has to recognize 2^n different scenarios, corresponding to the whole list of successes and failures. Of these, there are only $(n + 1)$ different levels of aggregate risk, corresponding to the total number of dotcomes that succeed or fail, and the probability distribution of aggregate risk is binomial if the risks are independent. Within each level of aggregate risk, there will be some lucky and some unlucky Geeks; that is individual risk. See the Cora-Ira example in the long handout "Financial Markets" to see how such risks are priced in the market.

Question 3: (35 points)

(a) (10 points) Owning a share of Hunters stock will give you the same pattern of payoffs in the three states of the worlds, namely \$100 in Animal, \$75 in Vegetable, and \$60 in Mineral, as you would get from a portfolio consisting of 100 Animal-state of the world Arrow-Debreu securities (ADSs for short), 75 Vegetable-state of the world ADSs, and 60 Mineral-state of the world ADSs. By absence of arbitrage in equilibrium, the share should have the same value as the equivalent portfolio. Similar equations apply to the other two companies' shares. Therefore

$$\begin{array}{rcl} 100 P_a + 75 P_v + 60 P_m & = & 70 \\ 80 P_a + 125 P_v + 60 P_m & = & 75 \\ 100 P_a + 50 P_v + 150 P_m & = & 110 \end{array}$$

Solving these,

$$P_a = 0.25, \quad P_v = 0.2, \quad P_m = 0.5$$

(b) (7 points) If you hold a package consisting of one ADS of each type, that will give you the entitlement to a dollar next year no matter which state of the world transpires, that is, a sure dollar. For this, today you will be willing to pay its discounted present value, namely $1/(1+r)$. Therefore

$$P_a + P_v + P_m = 1/(1+r)$$

or

$$r = 1/(P_a + P_v + P_m) - 1 = 1/0.95 - 1 = 1/19 = 0.0526, \quad \text{or} \quad 5.26 \text{ per cent}$$

(c) (10 points) If a package consisting of X_H shares of Hunters stock, X_G shares of Gatherers stock, and X_D shares of Diggers stock replicates one A-state of the world ADS, their payoff patterns must be equal in all state of the worlds, or

$$\begin{aligned} 100 X_H + 80 X_G + 100 X_D &= 1 \\ 75 X_H + 125 X_G + 50 X_D &= 0 \\ 60 X_H + 60 X_G + 150 X_D &= 0 \end{aligned}$$

Solving these, $X_H = 0.02561$, $X_G = -0.01341$ and $X_D = -0.00488$ (negative holdings indicate short selling, or issuing such a security). Then, by absence of arbitrage in equilibrium, the price of an A-state of the world ADS must equal the corresponding linear combination of the values of these stocks:

$$P_a = 70 X_H + 75 X_G + 110 X_D = 0.25$$

Similar calculations can be done for the other two ADSs. This method involves more algebra, but has the offsetting advantage of clarifying and fixing the idea and technique of constructing an equivalent (replicating) portfolio.

(d) (8 points) You will exercise your option in the Animal state of the world, paying 90 to get a stock that can immediately be resold for 100, thereby making a profit of $100 - 90 = 10$. Similarly you will exercise your option in the Mineral state of the world for a profit of $150 - 90 = 60$. You will let the option lapse in the Vegetable state of the world, so your profit in that state will be zero. Thus your option is exactly equivalent to a package consisting of 10 A-state of the world ADSs, 0 V-state of the world ADSs, and 60 M-state of the world ADSs. So the price P_O of the option today must be

$$P_O = 10 P_a + 0 P_v + 60 P_m = 10 \times 0.25 + 60 \times 0.5 = 2.5 + 30 = 32.5$$