ECO 317 – Economics of Uncertainty – Fall Term 2009
Problem Set 5 – DRAFT Answer Key

Question 1: (35 points)

(a) (2 points)

$$EU_i = \sum_{s=1}^{S} \pi_s \, U_i(C_{is})$$

(b) (2 points)

$$\sum_{i=1}^{I} C_{is} \leq C_s \quad \text{for all } s\,.$$

(c) (7 points)

$$\begin{aligned}
\mathcal{L} &= \sum_{i=1}^{I} EU_i + \sum_{s=1}^{S} \lambda_s \left\{ C_s - \sum_{i=1}^{I} C_{is} \right\} \\
&= \sum_{i=1}^{I} \left\{ \sum_{s=1}^{S} \pi_s \, U_i(C_{is}) \right\} + \sum_{s=1}^{S} \lambda_s \left\{ C_s - \sum_{i=1}^{I} C_{is} \right\} \\
&= \sum_{i=1}^{I} \sum_{s=1}^{S} \{ \pi_s \, U_i(C_{is}) - \lambda_s \, C_{is} \} + \sum_{s=1}^{S} \lambda_s \, C_s
\end{aligned}$$

(d) (7 points)

$$\pi_s \, U_i'(C_{is}) = \lambda_s \quad \text{for all } i \text{ and } s\,.$$

(e) (7 points) If (C_{is}^*) is optimal for given total amounts (C_s), then for consumers 1 and 2 and any state s

$$\pi_s \, U_1'(C_{1s}^*) = \pi_s \, U_2'(C_{2s}^*) \quad (= \lambda_s)$$

If $(2\,C_{is}^*)$ is optimal when the total amounts are $(2\,C_s)$, then for the same two consumers and state s,

$$\pi_s \, U_1'(2\,C_{1s}^*) = \pi_s \, U_2'(2\,C_{2s}^*) \quad (= \nu_s)$$

where ν_s is the Lagrange multiplier for the state-s constraint with the new doubled total amounts. Dividing the second equation by the first gives

$$\frac{U_1'(2\,C_{1s}^*)}{U_1'(C_{1s}^*)} = \frac{U_2'(2\,C_{2s}^*)}{U_2'(C_{2s}^*)}\,. \tag{1}$$

(f) (10 points)

If all consumers have identical utility-of-consequences functions U_i, say U, then the first-order conditions for state s become

$$\pi_s \, U'(C_{is}) = \lambda_s \quad \text{for all } i\,.$$

Then C_{is} must be equal for all i, and so each equal to C_s/I. The same argument holds when the total amounts are doubled, so all consumer's allocations become $2\,C_s/I$, and (1) holds, with each side equal to $U'(2\,C_s/I)/U'(C_s/I)$.

If they have different U_i functions but have the same constant relative risk aversion coefficient ρ, then

$$U_i'(C) = k_i \; C^{-\rho}$$

for all i, where the k_i are constants which may be different for different consumers. Then (1) holds, with each side equal to $2^{-\rho}$.

If they have different U_i functions but with the same constant absolute risk-aversion coefficient α, then

$$U_i'(C) = k_i \; e^{-\alpha C}$$

and (1) becomes

$$\exp(-\alpha \, C_{1s}^*) = \exp(-\alpha \, C_{2s}^*), \qquad \text{or} \qquad C_{1s}^* = C_{2s}^* \,,$$

which does not in general hold.

Question 2: (45 points)

(a) (2 points)

$$\begin{aligned} EU_X &= \sum_{s=1}^{S} \pi_s \; F(X_s) = \sum_{s=1}^{S} \pi_s \; F(x_s \, W_s) \\ EU_Y &= \sum_{s=1}^{S} \pi_s \; G(Y_s) = \sum_{s=1}^{S} \pi_s \; G(\,(1-x_s)\,W_s)\,. \end{aligned}$$

(b) (5 points)

$$\begin{aligned} \mathcal{L} &= \sum_{s=1}^{S} \pi_s \; F(x_s \, W_s) + \lambda \; \left\{ \sum_{s=1}^{S} \pi_s \; G(\,(1-x_s)\,W_s) - k \; \right\} \\ &= \sum_{s=1}^{S} \pi_s \; [\; F(x_s \, W_s) + \lambda \; G(\,(1-x_s)\,W_s) \;] - \lambda \, k \end{aligned}$$

(c) (6 points)

$$\pi_s \, W_s \; F'(x_s \, W_s) = \lambda \, \pi_s \, W_s \; G'(\,(1-x_s)\,W_s) \qquad \text{for all } s$$

or

$$F'(x_s \, W_s) = \lambda \; G'(\,(1-x_s)\,W_s) \qquad \text{for all } s \tag{2}$$

(d) (5 points) If Yvonne is risk-neutral, then $G'(Y) = g$, a constant, for all Y, and (2) yields

$$F'(x_s \, W_s) = \lambda \; k \qquad \text{for all } s$$

Because Xavier is strictly risk-averse ($F'' < 0$), this defines a unique $x_s \, W_s$, the same, say h, for all states s. Hence $x_s = h \,/\, W_s$ for all s, where h is a constant. But the simpler interpretation is that Xavier's total consumption X_s is the same in all states. When Yvonne is risk-neutral, it is efficient for her to bear all the risk.

(e) (5 points) If the two have the same constant relative risk-aversion coefficient ρ, then (2) becomes

$$(x_s\, W_s)^{-\rho} = \lambda\, (\, (1 - x_s)\, W_s)^{-\rho}$$

or

$$x_s = \lambda^{-1/\rho}\, (1 - x_s)$$

so x_s is the same for all s.

(f) (9 points) If the utility-of-consequences functions are

$$U_X(X_s) = -\, 1/X_s, \qquad U_Y(Y_s) = \ln(Y_s)\,,$$

then (2) becomes

$$(x_s\, W_s)^{-2} = \lambda\, (\, (1 - x_s)\, W_s)^{-1}\,,$$

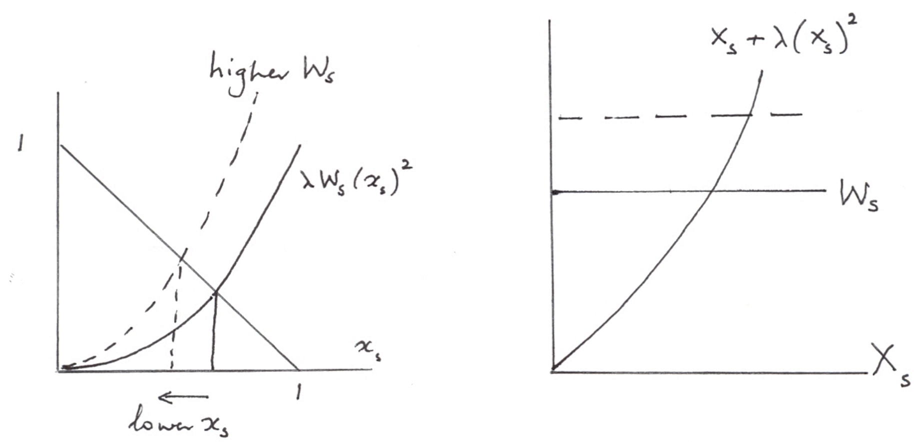
which simplifies to

$$\lambda\, W_s\, (x_s)^2 = 1 - x_s\,.$$

The sketch on the left below shows that Xavier gets a smaller fraction x_s in states with higher total W_s. Multiplying the above equation by W_s we see that his total amount $X_s = x_s\, W_s$ satisfy

$$X_s + \lambda\, (X_s)^2 = W_s$$

The right hand side shows the sketch of this; he gets a larger absolute amount X_s in states with higher total W_s.



(g) (6 points) If Xavier has constant relative risk aversion ρ and Yvonne has constant relative risk aversion θ, then (2) expressed in terms of total allocations becomes

$$(X_s)^{-\rho} = a\lambda\, (Y_s)^{-\theta}$$

where a is another constant (the F and the G functions might have different constant factors in them), or

$$X_s = (a\lambda)^{-1/\rho}\ (Y_s)^{\theta/\rho}$$

As ρ goes to infinity, Y_s remains bounded and therefore $(Y_s)^{\theta/\rho}$ tends to 1. (Extra note: λ changes as ρ changes, so the first factor need not go to 1.) Therefore X_s is the same for all states.

Question 3: (20 points)

Consider the problem of constrained Pareto efficient allocation using shares alone, from Note 13 (or Slides 13). Take the table on p. 5 of the note or slides.

(a) (14 points) The means are

$$\begin{aligned}\mu_A &= \tfrac{1}{4}\ [30(1-\phi)+40(1-\psi)] + \tfrac{1}{4}[10(1-\phi)+40(1-\psi)] + \tfrac{1}{4}\ [30(1-\phi)] + \tfrac{1}{4}\ [10(1-\psi)] \\ &= 20\ (1-\phi)+20(1-\psi)\end{aligned}$$

Similarly

$$\mu_B = 20\ \phi + 20\ \psi$$

Variances

$$\begin{aligned}V_A &= E[(X_A)^2] - (E[X_A])^2 \\ &= \tfrac{1}{4}\ [30(1-\phi)+40(1-\psi)]^2 + \tfrac{1}{4}[10(1-\phi)+40(1-\psi)]^2 \\ &\quad + \tfrac{1}{4}\ [30(1-\phi)]^2 + \tfrac{1}{4}\ [10(1-\phi)]^2 - [20\ (1-\phi)+20(1-\psi)]^2 \\ &= (225+25+225+25-400)\,(1-\phi)^2 + (400+400-400)\,(1-\psi)^2 \\ &\quad + (600+200-800)\,(1-\phi)(1-\psi) \\ &= 100\,(1-\phi)^2 + 400\,(1-\psi)^2\end{aligned}$$

Similarly

$$V_B = 100\ \phi^2 + 400\ \psi^2$$

(b) (3 points) Therefore

$$\begin{aligned}MV_A &= 20\ (1-\phi)+20(1-\psi) - \tfrac{1}{5}\ \left[100\,(1-\phi)^2 + 400\,(1-\psi)^2\right] \\ &= 20\ (1-\phi)+20(1-\psi) - 20\,(1-\phi)^2 - 80\,(1-\psi)^2\end{aligned}$$

and

$$\begin{aligned}MV_B &= 20\ \phi + 20\ \psi - \tfrac{1}{20}\ \left[100\ \phi^2 + 400\ \psi^2\right] \\ &= 20\ \phi + 20\ \psi - 5\ \phi^2 - 20\ \psi^2\end{aligned}$$

(c) (2 points) For $\phi = \psi = 0.6$ we have

$$MV_A = 20*0.4 + 20*0.4 - 20*0.16 - 80*0.16 = 0$$

and

$$MV_B = 20*0.6 + 20*0.6 - 5*0.36 - 20*0.36 = 24 - 9 = 15$$