ECO 317 – Economics of Uncertainty – Fall Term 2009
Problem Set 5 – Due December 3

Question 1: (35 points)

Do Exercise 10.3, p. 167 from Eeckhoudt-Gollier-Schlesinger. Use the following notation and steps of analysis.

Let the consumers be labelled $i = 1, 2 \ldots I$, and the states of the world $s = 1, 2 \ldots S$. Let pi_s denote the probability of state s. Let C_s denote the total amount of consumption (exogenously given) available in state s, and C_{is} the amount allocated (the planner's choice variables) to consumer i in state s. Let U_i denote consumer i's utility-of-consequences function. Assume all consumers are risk-averse.

(a) (2 points) Write down the expression for consumer i's expected utility EU_i.

(b) (2 points) What is the constraint in state s for the planner's allocation to be feasible?

(c) (7 points) The planner wants to maximize the sum of the expected utilities found in (a) above, subject to all the feasibility constraints for all states found in (b) above. Write down the Lagrangian for this optimization problem.

(d) (7 points) Find the first-order conditions for the planner's optimization problem.

(e) (7 points) Suppose that for a given set of total amounts (C_s), the allocation (C^*_{is}) is optimal. The question asks if $(2\, C^*_{is})$ will be optimal when the total amounts are $(2\, C_s)$. Show that if if it does, then for any two individuals, say 1 and 2, and any state s,

$$\frac{U'_1(2\, C^*_{1s})}{U'_1(C^*_{1s})} = \frac{U'_2(2\, C^*_{2s})}{U'_2(C^*_{2s})}\,.$$

(f) (10 points) Experiment with a few cases to find out when this can be so. What if all consumers have identical utility-of-consequences functions U_i? What if they have different U_i functions but have the same constant relative risk aversion? What if they have different U_i functions but with the same constant absolute risk-aversion?

Question 2: (45 points)

This question is also about a planner's Pareto efficient risk allocation. Note that to answer parts (d)–(g), you don't have to solve the constrained maximization problem completely (including solving out for the Lagrange multiplier), but can work from the first-order conditions leaving the Lagrange multiplier as it is.

There are numerous states of the world, labelled $s = 1, 2 \ldots S$. The probability of state s is π_s. The total available wealth in state s is denoted by W_s, and the states are labelled in increasing order of wealth, so $W_1 < W_2 \ldots < W_S$.

There are two risk averse consumers, Xavier Fernandez and Yvonne Gollier. Xavier's utility-of-consequences function is $F(X_s)$ and Yvonne's is $G(Y_s)$, where X_s and Y_s denote their respective wealth allocation (equals consumption) quantities in state s.

Suppose the planner allocates the fraction x_s of the total wealth in state s to Xavier and the fraction $(1 - x_s)$ to Yvonne.

(a) (2 points) Write down the expressions for the expected utilities EU_X and EU_Y of the two consumers.

(b) (5 points) Find the expression for the Lagrangian for the planner's problem of maximizing EU_X subject to $EU_Y \geq k$ where k is a constant.

(c) (6 points) Find the first-order conditions for the optimal choice of the fractions (x_s) in all the states.

(d) (6 points) If Yvonne is risk-neutral, while Xavier is strictly risk-averse, show that $x_s = h \,/\, W_s$ for all s, where h is a constant. Interpret this result.

(e) (6 points) If the two have the same constant relative risk-aversion, show that x_s is the same for all s.

(f) (9 points) If the utility-of-consequences functions defined over the total consumption amounts X_s and Y_s are

$$U_X(X_s) = -\,1/X_s, \qquad U_Y(Y_s) = \ln(Y_s)\,,$$

show that

$$\lambda\, W_s\, (x_s)^2 = 1 - x_s\,.$$

By sketching these two functions, show that Xavier gets a smaller fraction x_s in states with higher total W_s. What about his total amount $X_s = x_s\, W_s$?

(g) (6 points) If Xavier has constant relative risk aversion ρ and Yvonne has constant relative risk aversion θ, and $\rho \to \infty$ while θ stays constant and finite, show that in the limit Xavier's total allocation X_s is the same for all states s.

Question 3: (20 points)

Consider the problem of constrained Pareto efficient allocation using shares alone, from Note 13 (or Slides 13). Take the table on p. 5 of the note or slides.

(a) (14 points: 2 for each mean, 5 for each variance) Calculate the expressions for the means and variances of the allocations of each of the two people A and B, over the four states HH, HL, LH and LL, as functions of the shares ϕ and ψ.

(b) (4 points) Use these to find the expressions for the mean-variance objectives MV_A and MV_B.

(c) (2 points) Calculate MV_A and MV_B when $\phi = \psi = 0.6$.