ECO 317 – Economics of Uncertainty – Fall Term 2009
Problem Set 7 – Answer Key

The distribution was as follows:

90-99	80-89	< 80
6	3	1

Question 1: (60 points)

In this problem, the quantities supplied are observed by the government, so its mechanism can specify both the quantity and the money as functions of the reported type. That is, the contract says: choose from three alternatives: [1] supply Q_1 and receive M_1, [2] supply Q_2 and receive M_2, [3] supply Q_3 and receive M_3. Some of you treated the quantity as chosen by the firm, as if this was moral hazard. But the grading was relatively generous.

(a) (5 points) The expression for the profit of type-i BMC as a function of the quantity Q it supplies and the money M it receives is

$$\Pi_i = M - c_i\, Q$$

Along an iso-profit curve $\Pi_i =$ constant, the slope is

$$dM/dQ = c_i$$

Therefore in (Q, M) space, the iso-profit curves of successively higher cost type BMCs are steeper. This is single crossing.

(b) Incentive-compatibility conditions: (6 points)

$$M_1 - c_1\, Q_1 \;\geq\; M_2 - c_1\, Q_2 \qquad \text{for 1 not to mimic 2} \tag{1}$$
$$M_1 - c_1\, Q_1 \;\geq\; M_3 - c_1\, Q_3 \qquad \text{for 1 not to mimic 3} \tag{2}$$
$$M_2 - c_2\, Q_2 \;\geq\; M_1 - c_2\, Q_1 \qquad \text{for 2 not to mimic 1} \tag{3}$$
$$M_2 - c_2\, Q_2 \;\geq\; M_3 - c_2\, Q_3 \qquad \text{for 2 not to mimic 3} \tag{4}$$
$$M_3 - c_3\, Q_3 \;\geq\; M_1 - c_3\, Q_1 \qquad \text{for 3 not to mimic 1} \tag{5}$$
$$M_3 - c_3\, Q_3 \;\geq\; M_2 - c_3\, Q_2 \qquad \text{for 3 not to mimic 2} \tag{6}$$

Assuming IC's hold, the participation constraints are: (3 points)

$$M_1 - c_1\, Q_1 \;\geq\; 0 \tag{7}$$
$$M_2 - c_2\, Q_2 \;\geq\; 0 \tag{8}$$
$$M_3 - c_3\, Q_3 \;\geq\; 0 \tag{9}$$

(c) (3 points) Your expected net benefit, denoted by N say, is

$$N = \pi_1\, [\, 2\,(Q_1)^{1/2} - M_1\,] + \pi_2\, [\, 2\,(Q_2)^{1/2} - M_2\,] + \pi_3\, [\, 2\,(Q_3)^{1/2} - M_3\,] \tag{10}$$

(d) (12 points) Now we are using only the constraints (1), (4) and (9). The last immediately gives us the expression for the lower bound on M_3

$$M_3 \geq c_3\ Q_3 \tag{11}$$

Next combine (9) and (4) to obtain

$$M_2 \geq c_2\ Q_2 - c_2\ Q_3 + c_3\ Q_3 \tag{12}$$

Finally, combine all three to obtain

$$M_1 \geq c_1\ Q_1 - c_1\ Q_2 + c_2\ Q_2 - c_2\ Q_3 + c_3\ Q_3 \tag{13}$$

From the objective function (10), we see that to maximize expected benefit, for whatever choices of the quantities you make, the money amounts should be kept as small as possible. Therefore all three of the inequalities (13), (12), and (11) should hold as equalities. This in turn implies that the three original constraints (1), (4), and (9) should all be binding.

(e) (12 points) With equalities in the three selected constraints, to verify that the others are fulfilled, consider

Type 1 mimicking type 3: From (13) and (11) holding as equations, we have

$$M_1 - M_3 = c_1\ Q_1 - c_1\ Q_2 + c_2\ Q_2 - c_2\ Q_3$$

Therefore

$$\begin{aligned}
(M_1 - c_1\ Q_1) - (M_3 - c_1\ Q_3) &= M_1 - M_3 - c_1\ Q_1 + c_1\ Q_3 \\
&= c_1\ Q_1 - c_1\ Q_2 + c_2\ Q_2 - c_2\ Q_3 - c_1\ Q_1 + c_1\ Q_3 \\
&= (c_2 - c_1)\ (Q_2 - Q_3) > 0
\end{aligned}$$

because $c_2 > c_1$ and we are assuming $Q_2 > Q_3$.

Type 3 mimicking type 2: From (12) and (11) holding as equations, we have

$$M_3 - M_2 = c_3\ Q_3 - c_2\ Q_2 + c_2\ Q_3 - c_3\ Q_3$$

Therefore

$$\begin{aligned}
(M_3 - c_3\ Q_3) - (M_2 - c_3\ Q_2) &= M_3 - M_2 - c_3\ Q_3 + c_3\ Q_2 \\
&= c_3\ Q_3 - c_2\ Q_2 + c_2\ Q_3 - c_3\ Q_3 - c_3\ Q_3 + c_3\ Q_2 \\
&= -c_2\ Q_2 + c_2\ Q_3 - c_3\ Q_3 + c_3\ Q_2 \\
&= (c_3 - c_2)\ (Q_2 - Q_3) > 0 \text{ because } c_2 < c_3 \text{ and } Q_2 > Q_3.
\end{aligned}$$

Type 2 mimicking type 1: From (13) and (12) holding as equations, we have

$$M_2 - M_1 = -c_1\ Q_1 + c_1\ Q_2$$

Therefore

$$\begin{aligned}
(M_2 - c_2\ Q_2) - (M_1 - c_2\ Q_1) &= M_2 - M_1 - c_2\ Q_2 + c_2\ Q_1 \\
&= -c_1\ Q_1 + c_1\ Q_2 - c_2\ Q_2 + c_2\ Q_1 \\
&= (c_2 - c_1)\ (Q_1 - Q_2) > 0 \text{ because } c_1 < c_2 \text{ and } Q_1 > Q_2.
\end{aligned}$$

Type 3 mimicking type 1: We already have an expression for $M_1 - M_3$, so

$$\begin{aligned}
(M_3 - c_3\ Q_3) - (M_1 - c_3\ Q_1) &= M_3 - M_1 - c_3\ Q_3 + c_3\ Q_1 \\
&= -c_1\ Q_1 + c_1\ Q_2 - c_2\ Q_2 + c_2\ Q_3 - c_3\ Q_3 + c_3\ Q_1 \\
&= c_3\ (Q_1 - Q_3) - c_1\ (Q_1 - Q_2) - c_2\ (Q_2 - Q_3) \\
&= c_3\ (Q_1 - Q_2) + c_3\ (Q_2 - Q_3) - c_1\ (Q_1 - Q_2) - c_2\ (Q_2 - Q_3) \\
&= (c_3 - c_1)\,(Q_1 - Q_2) + (c_3 - c_2)\,(Q_2 - Q_3) > 0
\end{aligned}$$

Participation of type 2: From (12), now known to hold as an equation, we have

$$M_2 - c_2\ Q_2 = (c_3 - c_2)\ Q_3 > 0 \tag{14}$$

Participation of type 1: Similarly, from (13) we have

$$M_1 - c_1\ Q_1 = (c_2 - c_1)\ Q_2 + (c_3 - c_2)\ Q_3 > 0 \tag{15}$$

(f) (2 points) After substitution for the M_i from (13), (12), and (11) holding as equations, we have

$$\begin{aligned}
N &= 2\ [\,\pi_1\,(Q_1)^{1/2} + \pi_2\,(Q_2)^{1/2} + \pi_3\,(Q_3)^{1/2}\,] \\
&\quad -\pi_1\ [\,c_1\ Q_1 - c_1\ Q_2 + c_2\ Q_2 - c_2\ Q_3 + c_3\ Q_3\,] \\
&\quad -\pi_2\ [\,c_2\ Q_2 - c_2\ Q_3 + c_3\ Q_3\,] - \pi_3\ [\,c_3\ Q_3\,]
\end{aligned}$$

(g) (10 points) It remains to choose the Q_i to maximize this. The first order conditions are

$$\frac{\partial N}{\partial Q_1} = \pi_1\ [\,(Q_1)^{-1/2} - c_1\,] = 0 \tag{16}$$

$$\frac{\partial N}{\partial Q_2} = \pi_2\ [\,(Q_2)^{-1/2} - c_2\,] - \pi_1\ (c_2 - c_1) = 0 \tag{17}$$

$$\frac{\partial N}{\partial Q_3} = \pi_3\ [\,(Q_3)^{-1/2} - c_3\,] - (\pi_1 + \pi_2)\ (c_3 - c_2) = 0 \tag{18}$$

Therefore

$$Q_1 = (c_1)^{-2} \tag{19}$$

$$Q_2 = \left[\ c_2 + \frac{\pi_1}{\pi_2}\ (c_2 - c_1)\ \right]^{-2} \tag{20}$$

$$Q_3 = \left[\ c_3 + \frac{\pi_1 + \pi_2}{\pi_3}\ (c_3 - c_2)\ \right]^{-2} \tag{21}$$

(h) (5 points) From (20) and (19) we see that

$$Q_2 < (c_2)^{-2} < (c_1)^{-2} = Q_1$$

so this part does not need any further conditions. But from (21) and (20), for $Q_3 < Q_2$ we need

$$c_3 + \frac{\pi_1 + \pi_2}{\pi_3}\,(c_3 - c_2) > c_2 + \frac{\pi_1}{\pi_2}\,(c_2 - c_1)$$

or

$$\left[1 + \frac{\pi_1 + \pi_2}{\pi_3}\right]\,(c_3 - c_2) > \frac{\pi_1}{\pi_2}\,(c_2 - c_1)$$

or

$$\frac{1}{\pi_3}\,(c_3 - c_2) > \frac{\pi_1}{\pi_2}\,(c_2 - c_1)$$

or

$$\frac{c_3 - c_2}{c_2 - c_1} > \frac{\pi_1\,\pi_3}{\pi_2}$$

(i) (5 points) Relation of these results to the general principles of screening: need to share surplus for incentive compatibility, and the need to induce distortions to reduce the rent-sharing. Brief answers coming from you would suffice, but read the following more detailed statements carefully to review the theory:

[1] From (11) holding as an equation, we see that the worst type 3 gets zero excess profit, but the other two lower bounds (12) and (13) show that the other two types are given just enough surplus to eliminate their successive gains from pretending to have higher costs. In fact we can express these two as

$$M_2 - c_2\,Q_2 = (c_3 - c_2)\,Q_3$$

(so type 2 is given enough profit to offset the cost saving it would have made on its sales if it had pretended to be type 3), and

$$M_1 - c_1\,Q_1 = (c_2 - c_1)\,Q_2 + (c_3 - c_2)\,Q_3$$

(so type 1 is given enough profit to offset both the cost saving it would have made on its sales as type 2, plus the profit it would have been given for presenting itself as being type 2).

[2] So as not to give away too much surplus to types 1 and 2 for the truthful revelation, you find it optimal to reduce Q_2 and Q_3 below their ideal or first best levels Q_2^* and Q_3^* defined by setting the marginal benefit equal to marginal cost, namely

$$(Q_2^*)^{-1/2} = c_2, \qquad (Q_3^*)^{-1/2} = c_3$$

as can be seen by comparing these with the first order conditions (17) and (18) that define the information constrained optimal Q_2 and Q_3. There is no need to "distort" Q_1 in this way because there is no type with even lower cost who needs to be deterred from pretending to be type 1.

[3] The sizes of these distortions in the quantities purchased from each type depend on the relative numbers of that type versus the better types to whom surplus must be given away to deter them from pretending to be the type under consideration. Therefore if π_3 is much larger than $\pi_1 + \pi_2$, it is not optimal to distort Q_3 very much, whereas if within the first two types, π_2 is much smaller than π_1, it is optimal to distort Q_2 a lot. This threatens the inequality $Q_2 > Q_3$ on which the whole analysis was based. [Additional information: In this situation it is necessary to re-solve the problem, and it turns out that types 2 and 3 will be "bunched" at some common quantity $Q_{2,3}$ instead of being separated.]

So the optimal contract acts as a cost plus contract for the worst type. For the best type, the contract could effectively work like a fixed price contract at price c_1, but with an additional fixed sum payment of excess profit (equal to $(c_2 - c_1)\, Q_2 + (c_3 - c_2)\, Q_3$).

Question 2: (40 points)

Part (a):

(i) (4 points) Only ϵ is random in the expressions

$$w = h + s\,x + s\,\epsilon, \qquad y - w = -h + (1-s)\,x + (1-s)\,\epsilon\,.$$

Therefore

$$\mathrm{E}[w] = h + s\,x, \quad \mathrm{V}[w] = s^2\,v, \qquad \mathrm{E}[y-w] = -h + (1-s)\,x, \quad \mathrm{V}[y-w] = (1-s)^2\,v\,.$$

(ii) (2 points) Substituting,

$$U_P = -h + (1-s)\,x - \tfrac{1}{2}\,\alpha_P\,(1-s)^2\,v\,,$$

and

$$U_A = h + s\,x - \tfrac{1}{2}\,\alpha_A\,s^2\,v - \tfrac{1}{2}\,k\,x^2\,.$$

Part (b):

(i) (6 points) To maximize U_P subject to $U_A \geq U_A^0$, it is obviously optimal to keep h as low as possible. Therefore the agent's participation constraint will bind, and

$$-h = s\,x - \tfrac{1}{2}\,\alpha_A\,s^2\,v - \tfrac{1}{2}\,k\,x^2 - U_A^0\,.$$

This can be substituted into the expression for U_P. Then

$$\frac{\partial U_P}{\partial x} = s - k\,x + (1-s) = 0$$

yields $x = 1/k$, and

$$\frac{\partial U_A}{\partial s} = x - \alpha_A\,s\,v - x + \alpha_P\,(1-s)\,v = 0$$

yields

$$s = \frac{\alpha_P}{\alpha_P + \alpha_A}\,.$$

(ii) (4 points) In the mean-variance framework, both parties have constant absolute risk aversion, therefore zero wealth effects. The optimal risk-sharing is independent of their utility levels. The same share s achieves optimal risk-sharing, minimizing their total disutility from risk,

$$\tfrac{1}{2}\,\alpha_P\,(1-s)^2\,v + \tfrac{1}{2}\,\alpha_A\,s^2\,v$$

regardless of U_A^0. The U_A^0 affects the split of utilities between them and does so via the wage level h.

Part (c):

(i) (8 points) Maximizing U_A with respect to x for given h, s,

$$\frac{\partial U_A}{\partial x} = s - k\,x = 0 \qquad \text{yields} \qquad x = s/k\,.$$

(ii) (2 points) Substituting this expression for x into those for U_P and U_A:

$$U_P = -h + (1-s)\,\frac{s}{k} - \tfrac{1}{2}\,\alpha_P\,(1-s)^2\,v\,,$$

and

$$U_A = h + s\,\frac{s}{k} - \tfrac{1}{2}\,\alpha_A\,s^2\,v - \tfrac{1}{2}\,k\,\frac{s^2}{k^2}\,.$$

(iii) (10 points) Again it is optimal to keep h as low as possible. Therefore

$$-h = s\,\frac{s}{k} - \tfrac{1}{2}\,\alpha_A\,s^2\,v - \tfrac{1}{2}\,k\,\frac{s^2}{k^2} - U_A^0\,.$$

Then

$$\frac{\partial U_P}{\partial s} = \frac{2s}{k} - \alpha_A\,s\,v - \frac{s}{k} + \frac{1-2s}{k} + \alpha_P\,(1-s)\,v\,.$$

Setting this equal to zero and solving for s yields the given expression.

(iv) (4 points) The value of s when x is unverifiable is greater if

$$\frac{1+\alpha_P\,k\,v}{1+(\alpha_P+\alpha_A)\,k\,v} > \frac{\alpha_P}{\alpha_A+\alpha_P}\,.$$

This is true if and only if

$$(\alpha_A+\alpha_P) + (\alpha_A+\alpha_P)\,\alpha_P\,k\,v > \alpha_P + \alpha_P\,(\alpha_P+\alpha_A)\,k\,v\,,$$

or $\alpha_A > 0$ which is true.

The intuition is that with x unverifiable, s has a second useful role, namely it gives the agent an incentive to supply effort. For this reason the optimal s is bigger than it would be for reasons of optimal risk allocation alone.