12. ARROW-DEBREU MODEL OF GENERAL EQUILIBRIUM UNDER UNCERTAINTY

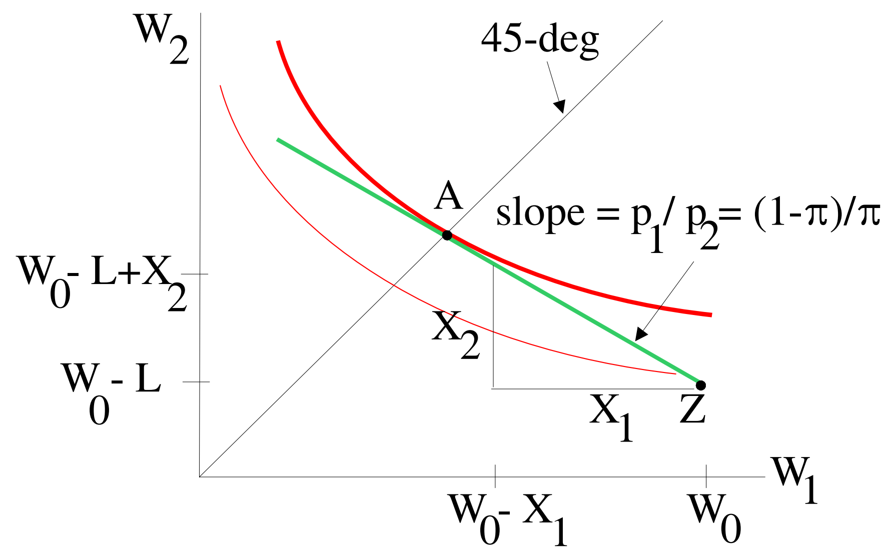
Usual insurance contract: Insured person pays premium p X in advance;
    company pays indemnity X if loss occurs, nothing otherwise
Equivalent alternative: Contract written in advance but no payments made in advance
    Insured pays company p X if loss does not occur (in state 1)
    Company pays insured (1-p) X if loss occurs (in state 2)
The trade (contract) made in advance is merely an exchange of promises
    Need governance mechanism for credibility, but otherwise no problem

Hence more general idea of "trade in contingent claims"
    Like betting slips – promises to pay specified money amounts or deliver specified goods
        if some specified state(s) of the world is(are) realized, and nothing otherwise
    Can pay a sure price in advance, or exchange it for another promise of equal market value

Examples: [1] Betting on sports events, racing bets, etc.
    [2] Betting on outcomes of political and economic events:
        Iowa electronic markets : http://www.biz.uiowa.edu/iem/
    [3] "Prediction markets," article at http://en.wikipedia.org/wiki/Prediction_market
    [4] "Trading in flu-tures," *The Economist,* Oct. 15, 2005
    [5] DARPA'S "Policy analysis market," so-called "terrorism futures market," now cancelled

DEMAND FOR INSURANCE

Objects traded are slips of paper that promise
S_1 : "\$1 if state 1" , S_2 : "\$1 if state 2"
Trading occurs before uncertainty is resolved
Prices \$$p_1$ for one slip S_1 ; \$$p_2$ for one slip S_2
Traders price-takers; probability of state 2 is π



Risk-averse insured person: will have wealth
W_0 in state 1 (no-loss), $W_0 - L$ in state 2 (loss)
(so p_2 is like premium for \$1 of indemnity)
Equivalently, has endowments of
W_0 of S_1-slips, $W_0 - L$ of S_2-slips
Wants to sell X_1 of S_1 slips, buy X_2 of S_2 slips
Budget constraint $p_1 X_1 - p_2 X_2 = 0$
if trade in these markets must be balanced
(Imbalance corresponds to saving or dissaving; will allow later.)
Objective: $EU = (1-\pi)\, U(W_0 - X_1) + \pi\, U(W_0 - L + X_2)$
FOCs: $(1-\pi)\, U'(W_0 - X_1) = \lambda\, p_1$, $\pi\, U'(W_0 - L + X_2) = \lambda\, p_2$

Risk-neutral insurance company that sells S_2 slips has expected profit = $p_2 - \pi$ on each slip
Competition achieves zero profit: $p_2 = \pi$. Similarly, $p_1 = 1 - \pi$

Then FOCs become $U'(W_0 - X_1) = \lambda$, $U'(W_0 - L + X_2) = \lambda$ so full insurance

ARBITRAGE

Can have markets in the S_1 , S_2 slips that pay \$1 in one state, nothing in the other

Can also have a combo slip S_c that pays \$1 no matter which state occurs

What is the price p_c of the S_c slip in the market for slips (before resolution of uncertainty)?

It must equal 1 if there is no significant time delay between buying/selling these contracts and the resolving of uncertainty

(If there is delay, then $p_c = 1/(1+r)$ where r is the riskless rate of interest; ignore for now.)

Must have $p_1 + p_2 = p_c = 1$, regardless of whether there are any risk-neutral traders

Argument: [1] If $p_1 + p_2 > p_c$, simultaneously buy one S_c and sell 1 each of S_1 , S_2

Net profit $p_1 + p_2 - p_c > 0$ earned right now and riskless

After uncertainty resolves, collect \$1 on the S_c , to pay off \$1 on S_1 or S_2 depending on state

As people compete to exploit this opportunity, they will bid down p_1 , p_2

[2] If $p_1 + p_2 < p_c$, simultaneously sell one S_c and buy 1 each of S_1 , S_2

Net profit $p_c - p_1 - p_2 > 0$ earned right now and riskless

After uncertainty resolves, collect \$1 on S_1 or S_2 depending on state, and pay off \$1 on S_c

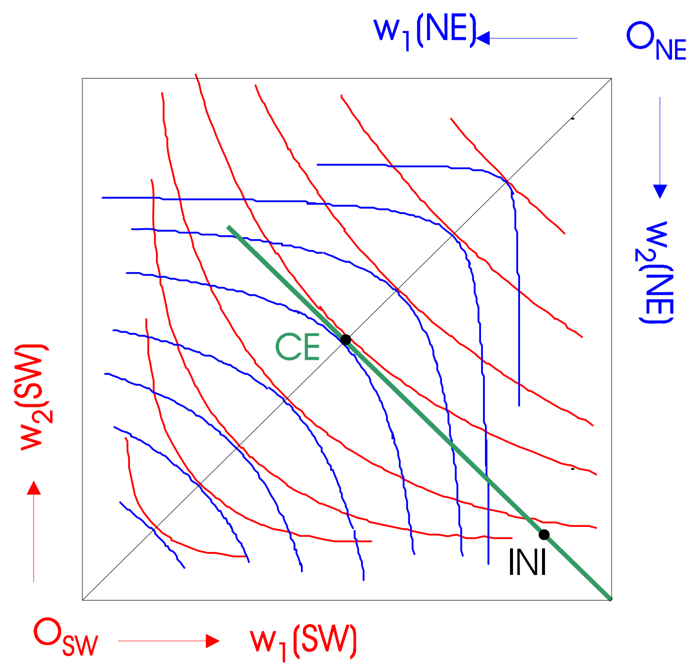
As people compete to exploit this opportunity, they will bid up p_1 , p_2

Arbitrage: purchasing a set of financial assets at a low price and selling an equivalent or repackaged set at a high price simultaneously. Arbitrageurs require no outlay of personal endowment; revenue generated from the selling contract pays off the costs of the buying contract and leaves a positive riskless net profit.

No-arbitrage principle: arbitrage opportunities cannot persist in equilibrium. This provides the basic method for establishing relationships among prices of different financial assets.

TRADE IN CONTINGENT CLAIMS WHEN BOTH SIDES ARE RISK-AVERSE

EXAMPLE 1 – NO AGGREGATE RISK



Total quantities of contingent claims
(on wealth, income, consumption as relevant)
are equal in the two states – box is square
Total $W_0 = W_G + W_B$ (G: good, B: bad)
Two people, SW, NE. Their risks are
perfectly negatively correlated
Initial endowments are
SW: (W_G , W_B), NE: (W_B , W_G)

SW's budget constraint is
$$p_1\, W_1(SW) + p_2\, W_2(SW) = p_1\, W_G + p_2\, W_B$$
He maxes
$$EU = (1-\pi)\, U_{SW}\, (W_1(SW)) + \pi\, U_{SW}(W_2(SW))$$
If prices are statistically fair: $p_1 = 1 - \pi$, $p_2 = \pi$
he will choose full insurance, demands: $W_1(SW) = W_2(SW) = (1 - \pi)\, W_G + \pi\, W_B$

Similarly, $W_1(NE) = W_2(NE) = (1 - \pi)\, W_B + \pi\, W_G$
Then, in state 1, total contingent claims $W_1(SW) + W_1(NE) = W_G + W_B = W_0$
Similarly in state 2. So fair prices yield competitive general equilibrium
Both traders are fully insured: each has the same wealth in the two states
but SW has more wealth in both states than does NE if $\pi < ½$;
conversely NE does better than SW if $\pi > ½$

EXAMPLE 2 – AGGREGATE RISK

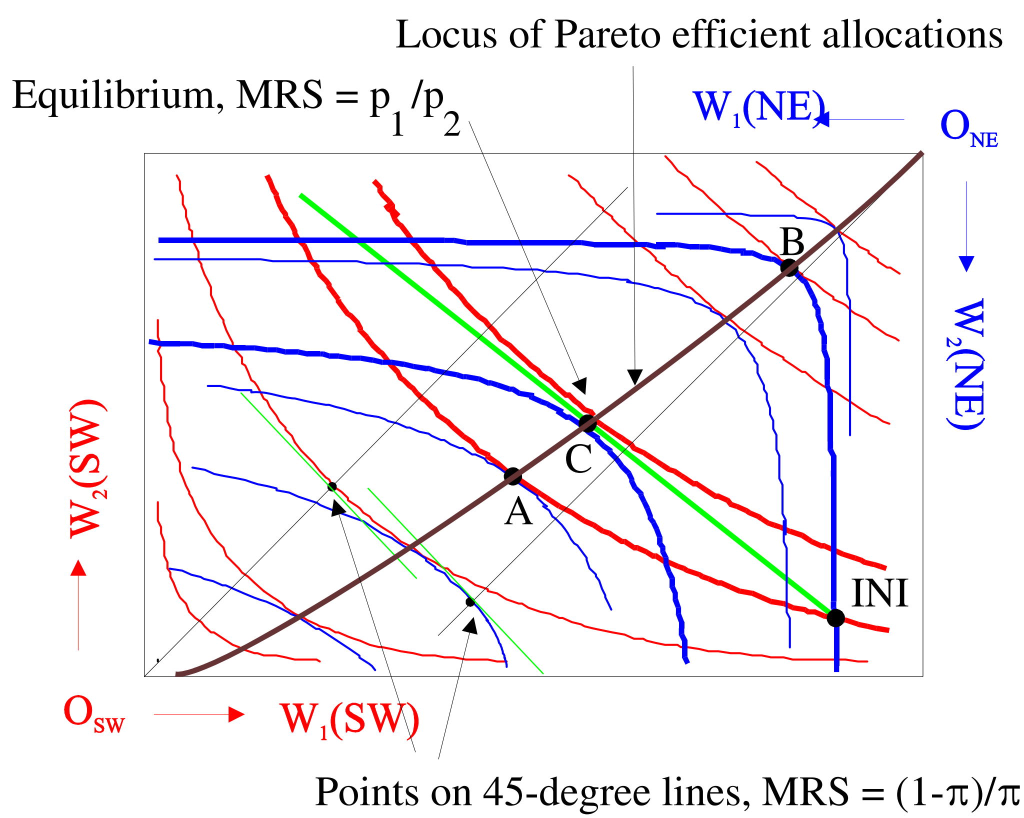
Total endowment $W_1 > W_2$: state 1 is "good" and state 2 is "bad"

SW is less risk-averse than NE
(ICs less sharply curved)
So equilibrium is closer to NE's
45-deg line than to SW's

At any efficient risk-allocation,
$p_1 / p_2 < (1-\pi)/\pi$
So $p_2 > \pi$ and $p_1 < (1-\pi)$
and $p_2 - \pi = (1-\pi) - p_1$
Costs more now to buy claim to
$1 in bad state than probability,
because both are risk-averse
and would want to buy at fair price

Today's value of whole market
$= p_1 W_1 + p_2 W_2$
$= (1 - \pi) W_1 + \pi W_2 - (p_2 - \pi) (W_1 - W_2)$
$< (1 - \pi) W_1 + \pi W_2$
So buying whole market today
yields excess expected return
This is aggregate risk premium; general equilibrium version of
the "price of market risk" of the mean-variance analysis in Handout 6 p. 10

INI = initial endowment, AB = core, C = equilibrium



SECURITIES, COMPLETE MARKETS, SPANNING

A contingent claim to \$1 in one state and nothing in any other state
is called an Arrow-Debreu security (ADS)

If there exist markets in Arrow-Debreu securities for all states, then
you can trade your initial ownership of contingent claims (ADSs), to obtain (consume) any other point in contingent claims space subject only to the budget constraint

More typically, objects traded are not pure ADSs, but securities
Each security is a specific combination of contingent claims

If there are enough of these, then ADSs for all states of the world
can be constructed as linear combinations of other available securities

Example to show when and how this can be done:

Two states of the world: 1 - oil price is high, 2 - oil price is low

Securities: share ownership in two firms, A - oil company, B - auto company

Value (dividend etc) of each share: A: \$2 in state 1, \$1 in state 2. B: \$1 in state 1, \$3 in state 2

Suppose you want a pure state-1 ADS. Try holding x of firm-A shares and y of firm-B shares

Need $2x + 1y = 1$; $1x + 3y = 0$. Solution: $x = 0.6$, $y = -0.2$

Exercise: similarly find the combination that recreates a pure state-2 ADS.

Corresponding pricing relations:

Suppose shares in the two firms have prices π_A, π_B respectively

What will be the prices P_1, P_2 of the ADSs?

No-arbitrage conditions: $\pi_A = 2 P_1 + 1 P_2$, $\pi_B = 1 P_1 + 3 P_2$

Solving, $P_1 = 0.6\,\pi_A - 0.2\,\pi_B$; exercise: find similar expression for P_2

GENERAL THEORY

States of the world: $s = 1$, 2, $\dots$ S
Prices (explicit or implicit) of pure Arrow-Debreu securities P_s

Firms' securities actually traded in markets: $f = 1$, 2, $\dots$ F
Firm f's security yields a_{fs} in state s

Can we construct pure ADSs for each state as
linear combinations of actually traded securities?
Do there exist X_{sf} such that,

$$\sum_{f=1}^{F} X_{sf}\ a_{fs'} = \begin{cases} 1 & \text{if } s' = s \\ 0 & \text{if } s' \neq s \end{cases}$$

(Negative Xs are OK; they correspond to short sales.)

Answer: if the matrix (a_{fs}) has rank S
i.e. the traded securities' payoff vectors that *span* the state space
Then we say that there is a complete set of financial markets

Prices of firms' securities Π_f relate to prices P_s of ADSs
by the no-arbitrage conditions of market equilibrium:

$$\Pi_f = \sum_{s=1}^{S} a_{fs} \; P_s$$

So once we can price pure ADSs, we can also price any
new security with any given payoff pattern across states of world
Examples: options and other derivative securities
Vector of prices of pure ADSs is "pricing kernel"

Conversely: given Π_f determined in financial markets,
will these equations determine P_s uniquely?
If so, they become implicit prices of Arrow-Debreu securities
even if such pure securities are not actually traded.

Answer: again, if the matrix (a_{fs}) has rank S
i.e. the payoff vectors of traded securities to *span* the state space
If $F > S$, can use submatrix of rank S to create ADSs
and then use no-arbitrage condition to price remaining $(F - S)$

Finance = General Equilibrium + Linear Algebra

Four-Scenario Example

Two farmers. Cora has COnstant (relative) Risk Aversion:

$$U(C) = \frac{1}{1-\rho}\ C^{1-\rho}$$

Ira has Infinite Risk Aversion. Output of each farmer can be either 1 or 2 with equal probability; independent. Four "states" with probability $\frac{1}{4}$ each:

g – "good state" – each has output 2; total output 4. b – "bad state" – each has 1; total 2.

c – Cora has 2 and Ira hast 1; total 3. i – Cora has 1 and Ira has 2; total 3.

Cora's budget constraint: $P_g\ C_g^c + P_c\ C_c^c + P_i\ C_i^c + P_b\ C_b^c = 2\ P_g + 2\ P_c + P_i + P_b$

Ira's budget constraint is $P_g\ C_g^i + P_c\ C_c^i + P_i\ C_i^i + P_b\ C_b^i = 2\ P_g + P_c + 2\ P_i + P_b$

Equilibrium conditions: total demands must equal the total outputs in each state:

$$C_g^c + C_g^i = 4,\ C_c^c + C_c^i = 3,\ C_i^c + C_i^i = 3,\ C_b^c + C_b^i = 2$$

We can find three relative prices using any three of these equations. Numerical solution:

Cora's Risk–Aversion Coefficient ρ	Cora's Consumption Quantities in Scenarios				Ira's Consumption Quantities (all Scenarios)	Prices of Arrow-Debreu Securities in Scenarios			
	g	c	i	b		g	c	i	b
0.001	2.50	1.50	1.50	0.50	1.50	0.99	1.00	1.00	1.01
0.50	2.60	1.60	1.60	0.60	1.40	0.78	1.00	1.00	1.64
1.00	2.68	1.68	1.68	0.68	1.32	0.63	1.00	1.00	2.44
2.00	2.81	1.81	1.81	0.81	1.19	0.51	1.00	1.00	4.99
10.00	2.99	1.99	1.99	0.99	1.01	0.02	1.00	1.00	1013