ECO 317 – Economics of Uncertainty – Fall Term 2009
Slides to accompany
21. INCENTIVES FOR EFFORT - MULTI-DIMENSIONAL CASES

1. The General Linear-Quadratic Framework

Notation:

$$\begin{aligned} \mathbf{x} &= (x_j),\ n\text{-vector of agent's effort (tasks), private information} \\ \mathbf{y} &= (y_i),\ m\text{-vector of principal's outcomes, verifiable} \\ w &= \text{agent's total compensation} \end{aligned}$$

Linear production function:

$$\mathbf{y} = \mathbf{M}\,\mathbf{x} + \mathbf{e} \qquad \text{or} \qquad y_i = \sum_{i=1}^{n} M_{ij}\; x_j + e_i\,, \tag{1}$$

where

$$\begin{aligned} \mathbf{M} &= M_{ij} = \partial y_i / \partial x_j\,, m\text{-by-}n \text{ matrix marginal products} \\ \mathbf{e} &= (e_i) m\text{-vector of normal random error or noise} \\ & \quad \text{zero mean, pos-sem-def variance-covariance matrix } \mathbf{V}. \end{aligned}$$

Linear compensation function:

$$w = h + \mathbf{s}'\,\mathbf{y} \tag{2}$$

where

$$\begin{aligned} h &= \text{fixed component} \\ \mathbf{s} &= m\text{-vector of marginal incentive bonus coefficients} \end{aligned}$$

Agent's objective (utility, or payoff):

$$U_A = \mathrm{E}[w] - \tfrac{1}{2}\,\alpha\,\mathrm{Var}[w] - \tfrac{1}{2}\,\mathbf{x}'\,\mathbf{K}\,\mathbf{x} \tag{3}$$

where

$$\begin{aligned} \alpha &= \text{agent's coefficient of constant absolute risk aversion} \\ \mathbf{K} &= n\text{-by-}n \text{ symmetric pos semi-def matrix} \end{aligned}$$

Two tasks i and j are substitutes if $K_{ij} > 0$ (increase in x_i raises marginal disutility of x_j and vice versa), and complements if $K_{ij} < 0$. Agent's outside opportunity utility U_A^0.

Principal's objective (utility, or payoff):

$$U_P = \mathrm{E}[\mathbf{p}'\,\mathbf{y} - w] \tag{4}$$

where $\mathbf{p}$ is m-vector of unit valuations.

2. One Principal, One Agent

We have

$$w = h + \mathbf{s}'\,\mathbf{y} = h + \mathbf{s}'\,\mathbf{M}\,\mathbf{x} + \mathbf{s}'\,\mathbf{e}$$

Therefore

$$\mathrm{E}[w] = h + \mathbf{s}'\;\mathbf{M}\,\mathbf{x}\,, \qquad \mathrm{Var}[w] = \mathbf{s}'\,\mathbf{V}\,\mathbf{s}$$

and

$$U_A = h + \mathbf{s}'\,\mathbf{M}\,\mathbf{x} - \tfrac{1}{2}\,\alpha\;\mathbf{s}'\,\mathbf{V}\,\mathbf{s} - \tfrac{1}{2}\;\mathbf{x}'\,\mathbf{K}\,\mathbf{x} \tag{5}$$

The agent chooses $\mathbf{x}$ to maximize this. The vector FOC (MAT203)

$$\mathbf{M}'\,\mathbf{s} - \mathbf{K}\,\mathbf{x} = 0\,. \tag{6}$$

SOC: $-\mathbf{K}$ negative semi-definite, which is true.

FOC using MAT 201 methods:

$$U_A = h + \sum_{i=1}^{m} \sum_{j=1}^{n} \; s_i \; M_{ij} \; x_j - \tfrac{1}{2} \; \sum_{i=1}^{m} \sum_{k=1}^{m} \; s_i \; V_{ik} \; s_k - \tfrac{1}{2} \; \sum_{h=1}^{n} \sum_{j=1}^{n} \; x_h \; K_{hj} \; x_j \; .$$

For any one component of $\mathbf{x}$, say x_g, we have

$$\frac{\partial U_A}{\partial x_g} = \sum_{i=1}^{m} \; s_i \; M_{ig} - \tfrac{1}{2} \; \sum_{h=1}^{n} \; x_h \; K_{hg} - \tfrac{1}{2} \; \sum_{j=1}^{n} \; K_{gj} \; x_j \; .$$

Rearranging and collecting terms into vector and matrices yields (6). In the process, the matrices $\mathbf{M}$ and $\mathbf{K}$ have to be transposed, and you need to remember that the latter is symmetric.

Solving (6) for $\mathbf{x}$:

$$\mathbf{x} = \mathbf{K}^{-1} \; \mathbf{M}' \; \mathbf{s} \quad \text{Agent's IC} \tag{7}$$

$$\mathrm{E}[\mathbf{y}] = \mathbf{M} \; \mathbf{K}^{-1} \; \mathbf{M}' \; \mathbf{s} \equiv \mathbf{N} \; \mathbf{s} \, ,$$

where

$$\mathbf{N} \;\; = \;\; \mathbf{M} \; \mathbf{K}^{-1} \; \mathbf{M}' \; m\text{-by-}m \text{ matrix of marginal products of bonus coefficients on principal's outcomes}$$

Agent's maximized or indirect utility function:

$$\begin{aligned} U_A^* &= h + \mathbf{s}'\,\mathbf{M}\,[\,\mathbf{K}^{-1}\,\mathbf{M}'\,\mathbf{s}] - \tfrac{1}{2}\,\alpha\;\mathbf{s}'\,\mathbf{V}\,\mathbf{s} - \tfrac{1}{2}\,[\,\mathbf{K}^{-1}\,\mathbf{M}'\,\mathbf{s}]'\,\mathbf{K}\,[\,\mathbf{K}^{-1}\,\mathbf{M}'\,\mathbf{s}] \\ &= h + \tfrac{1}{2}\;\mathbf{s}'\,\mathbf{M}\,\mathbf{K}^{-1}\,\mathbf{M}'\,\mathbf{s} - \tfrac{1}{2}\,\alpha\;\mathbf{s}'\,\mathbf{V}\,\mathbf{s} \\ &= h + \tfrac{1}{2}\;\mathbf{s}'\,\mathbf{N}\,\mathbf{s} - \tfrac{1}{2}\,\alpha\;\mathbf{s}'\,\mathbf{V}\,\mathbf{s} \end{aligned} \tag{8}$$

Principal's indirect utility function

$$\begin{aligned} U_P &= \mathbf{p}'\;\mathrm{E}[\mathbf{y}] - \mathrm{E}[w] \\ &= \mathbf{p}'\,\mathbf{N}\,\mathbf{s} - h - \mathbf{s}'\,\mathbf{M}\;[\,\mathbf{K}^{-1}\,\mathbf{M}'\,\mathbf{s}\,] \\ &= \mathbf{p}'\,\mathbf{N}\,\mathbf{s} - h - \mathbf{s}'\,\mathbf{N}\,\mathbf{s}\,. \end{aligned} \tag{9}$$

Use agent's binding PC $U_A^* = U_A^0$ to solve for h. Then

$$\begin{aligned} U_P &= \mathbf{p}'\,\mathbf{N}\,\mathbf{s} - U_A^0 + \tfrac{1}{2}\;\mathbf{s}'\,\mathbf{N}\,\mathbf{s} - \tfrac{1}{2}\,\alpha\;\mathbf{s}'\,\mathbf{V}\,\mathbf{s} - \mathbf{s}'\,\mathbf{N}\,\mathbf{s} \\ &= \mathbf{p}'\,\mathbf{N}\,\mathbf{s} - U_A^0 - \tfrac{1}{2}\;\mathbf{s}'\,\mathbf{N}\,\mathbf{s} - \tfrac{1}{2}\,\alpha\;\mathbf{s}'\,\mathbf{V}\,\mathbf{s} \end{aligned} \tag{10}$$

Maximize this w.r.t. $\mathbf{s}$, FOC is

$$\mathbf{N}\,\mathbf{p} - [\,\mathbf{N} + \alpha\,\mathbf{V}\,]\;\mathbf{s} = 0\,. \tag{11}$$

SOC: $(\mathbf{N} + \alpha\ \mathbf{V})$ positive semi-definite, which is true. Solving

$$\mathbf{s} = [\,\mathbf{N} + \alpha\,\mathbf{V}\,]^{-1}\ \mathbf{N}\ \mathbf{p}\,. \tag{12}$$

Check 1-dimensional special case: $\mathbf{p} = 1$, $\mathbf{M} = 1$, $\mathbf{K} = k$, and $\mathbf{V} = v$.
Then $\mathbf{N} = 1/k$, and (12) becomes

$$s = [(1/k) + \alpha\, v\,]^{-1}\ (1/k) = \frac{1}{1 + \alpha\, v\, k}\,.$$

3. One Task, Two Outcome Measures

$m = 2$ and $n = 1$. Make $\mathbf{M}$ (now a 2-by-1 column vector) $= (1, 1)'$ by choice of units. Let the matrix $\mathbf{V}$ be diagonal,

$$\mathbf{V} = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}; \quad \text{then} \quad \mathbf{N} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{k} \begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and

$$\mathbf{N} + \alpha\, \mathbf{V} = \frac{1}{k} \begin{pmatrix} 1 + k\,\alpha\, v_1 & 1 \\ 1 & 1 + k\,\alpha\, v_2 \end{pmatrix}.$$

Therefore

$$\mathbf{s} = k \begin{pmatrix} 1 + k\,\alpha\, v_1 & 1 \\ 1 & 1 + k\,\alpha\, v_2 \end{pmatrix}^{-1} \frac{1}{k} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{p}\,.$$

This simplifies to

$$s_1 = \frac{v_2\ (p_1 + p_2)}{v_1 + v_2 + k\,\alpha\, v_1\, v_2}, \qquad s_2 = \frac{v_1\ (p_1 + p_2)}{v_1 + v_2 + k\,\alpha\, v_1\, v_2}.$$

Implications:

[1] Even if $p_2 = 0$, $s_2 \neq 0$. Depends only on $(p_1 + p_2)$. Outcome 2 is useful because of its information content. Observe $s_1/s_2 = v_2/v_1$. If v_1 is large, rely on s_2.

[2]Even if $\alpha = 0$, each of s_1 and s_2 is $< (p_1 + p_2)$. Specifically,

$$s_1 = \frac{v_2\ (p_1 + p_2)}{v_1 + v_2}\,, \qquad s_2 = \frac{v_1\ (p_1 + p_2)}{v_1 + v_2}\,.$$

Thus $s_1 + s_2 = p_1 + p_2$. Total incentive has full power, but its split optimizes information.

4. Many tasks, Two Outcomes

Dimension n of $\mathbf{x}$ can be large. Dimension m of outcomes = 2. Take $p_2 = 0$ so outcome 2 has only information role. Suppose constraint $s_1 = 0$ so compensation must be based on outcome 2. How useful is it? Take $\alpha = 0$ to remove the issue of the agent's risk aversion.

Also assume $\mathbf{K}$ diagonal, $k\,\mathbf{I}_n$, to remove issue of effort interaction.

$$\begin{aligned} U_P &= (\,p_1\ \ 0\,) \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} 0 \\ s_2 \end{pmatrix} - \tfrac{1}{2}\ (\,0\ \ s_2\,) \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} 0 \\ s_2 \end{pmatrix} \\ &= p_1\, N_{12}\, s_2 - \tfrac{1}{2}\, N_{22}\, (s_2)^2 \,. \end{aligned}$$

The first-order condition (11) for s_2 (the only relevant component of $\mathbf{s}$) becomes

$$N_{22}\ s_2 = N_{12}\ p_1 \,.$$

Also

$$\mathbf{N} = \mathbf{M}\ (k\,\mathbf{I}_n)^{-1}\ \mathbf{M}' = \frac{1}{k}\ \mathbf{M}\,\mathbf{M}' \,.$$

Therefore

$$N_{22} = \frac{1}{k} \sum_{j=1}^{n} (M_{2j})^2 \,, \qquad N_{12} = \frac{1}{k} \sum_{j=1}^{n} M_{1j}\, M_{2j} \,,$$

and then

$$s_2 = \frac{\sum_{j=1}^{n} M_{1j}\, M_{2j}}{\sum_{j=1}^{n} (M_{2j})^2}\, p_1 \,.$$

The sign and magnitude of s_2 depend importantly on those of the numerator. This is inner product or covariance between the marginal effects of the actions on the two dimensions of outcome. (A large negative alignment would be just as valuable as a large positive alignment, making s_2 big and negative. Zero alignment makes indicator 2 useless.)

5. Substitutes and Complements in Efforts

Focus on effect of non-zero off-diagonal entries in $\mathbf{K}$. Get rid of all else:

[1] $m = n = 2$, [2] $p_1 = p_2 = p$,
[3] $\mathbf{M}$ diagonal $= \mathbf{I}_2$ by choice of units.
[4] $\mathbf{V}$ also diagonal $= v\,\mathbf{I}_2$, [5] the disutility matrix is

$$\mathbf{K} = \begin{pmatrix} k & \theta\,k \\ \theta\,k & k \end{pmatrix} .$$

where $k > 0$ and $-1 < \theta < 1$. Actions substitutes if $\theta > 0$,
complements if $\theta < 0$.

Then

$$\mathbf{x} = \mathbf{K}^{-1}\ \mathbf{I}_2\ \mathbf{s} = \mathbf{K}^{-1}\ \mathbf{s}\,,$$

so

$$
\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} k & \theta\,k \\ \theta\,k & k \end{pmatrix}^{-1} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \\
&= \frac{1}{k} \begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix}^{-1} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \\
&= \frac{1}{k\,(1-\theta^2)} \begin{pmatrix} 1 & -\,\theta \\ -\,\theta & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \\
&= \frac{1}{k\,(1-\theta^2)} \begin{pmatrix} s_1 - \theta\,s_2 \\ s_2 - \theta\,s_1 \end{pmatrix} \qquad (13)
\end{aligned}
$$

This is another view of the substitutes / complements distinction. The two are equivalent in the 2-by-2 case but can differ in higher dimensions.

Now $\mathbf{N} = \mathbf{I}_2\, \mathbf{K}^{-1}\, \mathbf{I}_2 = \mathbf{K}^{-1}$, so FOC (11) becomes

$$[\, \mathbf{K}^{-1} + \alpha\, v\, \mathbf{I}_2 \,]\ \mathbf{s} = \mathbf{K}^{-1}\ \mathbf{p}\,.$$

Premultiplying by $\mathbf{K}$ gives

$$[\, \mathbf{I}_2 + \alpha\, v\, \mathbf{K} \,]\ \mathbf{s} = \mathbf{p}\,,$$

or

$$\begin{pmatrix} 1 + \alpha\, v\, k & \theta\, \alpha\, v\, k \\ \theta\, \alpha\, v\, k & 1 + \alpha\, v\, k \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} p \\ p \end{pmatrix}.$$

This yields the solution

$$s_1 = s_2 = \frac{p}{1 + (1 + \theta)\, \alpha\, v\, k}\,.$$

Substitutes ($\theta > 0$), make it necessary to reduce the power of incentives on both outcomes, because sharpening the incentives on either will cause the agent to divert his effort away from the other. Conversely, complements ($\theta < 0$) enable strengthening of incentives to both tasks.

Implications for organization theory: when grouping tasks into departments, group together complements. Think of universities, IRS, Homeland Security etc. in this perspective.

6. Multiple Principals – Common Agency

Two actions and two outcomes ($m = n = 2$).
Production function $\mathbf{M} = \mathbf{I}_2$; so each action affects only one outcome.
Disutility of effort $\mathbf{K} = k\,\mathbf{I}_2$, so no substitutes/complement issue.

$\mathbf{N} = \mathbf{I}_2 \, \frac{1}{k} \, \mathbf{I}_2 \, \mathbf{I}_2 = \frac{1}{k} \, \mathbf{I}_2$

Error variance matrix $\mathbf{V} = v\,\mathbf{I}_2$.

Two principals. Each cares about only one outcome.

If they jointly implement the incentive scheme,

$$s_1 = \frac{p_1}{1 + \alpha\, v\, k}, \qquad s_2 = \frac{p_2}{1 + \alpha\, v\, k} \tag{14}$$

We want non-cooperative Nash equilibrium of principals' individual choices.

So now let the two principals act independently, offer respectively

$$w_1 = h_1 + s_{1,1}\; y_1 + s_{1,2}\; y_2 \tag{15}$$

$$w_2 = h_2 + s_{2,1}\; y_1 + s_{2,2}\; y_2 \tag{16}$$

Note: each includes the other's outcome to affect the agent's action.

Agent's total compensation

$$w = h + s_1 \; y_1 + s_2 \; y_2$$

where

$$h = h_1 + h_2, \qquad s_1 = s_{1,1} + s_{2,1}, \qquad s_2 = s_{1,2} + s_{2,2}$$

Then, from (7) the agent's effort choice is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/k & 0 \\ 0 & 1/k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

or $\mathbf{x} = \mathbf{s} \, / \, k$ or

$$x_1 = \frac{s_1}{k} = \frac{s_{1,1} + s_{2,1}}{k}, \quad x_2 = \frac{s_2}{k} = \frac{s_{1,2} + s_{2,2}}{k}$$

Then

$$
\begin{aligned}
\mathrm{E}[w_1] &= h_1 + s_{1,1}\ x_1 + s_{1,2}\ x_2 \\
&= h_1 + s_{1,1}\ \frac{s_1}{k} + s_{1,2}\ \frac{s_2}{k} \\
&= h_1 + \frac{1}{k}\ [\ s_{1,1}\,(s_{1,1} + s_{2,1}) + s_{1,2}\,(s_{1,2} + s_{2,2})\ ]
\end{aligned}
$$

Using the special forms of $\mathbf{N}$ and $\mathbf{V}$,

$$
\begin{aligned}
U_A^* &= h + \frac{1}{2k}\ \mathbf{s}'\,\mathbf{s} - \frac{1}{2}\,\alpha\,v\ \mathbf{s}'\,\mathbf{s} \\
&= h_1 + h_2 + \left[\frac{1}{2k} - \frac{\alpha\,v}{2}\right]\ \left[\ (s_{1,1} + s_{2,1})^2 + (s_{1,2} + s_{2,2})^2\ \right] \quad (17)
\end{aligned}
$$

Focus on principal 1.

$$U_P^1 = \mathrm{E}[\, p_1\, y_1 - w_1\,] = p_1\,\frac{s_{1,1}+s_{2,1}}{k} - h_1 - \frac{1}{k}\ \left[\ s_{1,1}\,(s_{1,1}+s_{2,1}) + s_{1,2}\,(s_{1,2}+s_{2,2})\ \right]$$

He chooses $(h_1, s_{1,1}, s_{1,2})$ to max this, given principal 2's $(h_2, s_{2,1}, s_{2,2})$ and respecting the agent's participation constraint $U_A^* \geq U_A^0$. So

$$h_1 = U_A^0 - h_2 - \left[\frac{1}{2k} - \frac{\alpha\, v}{2}\right]\ \left[\ (s_{1,1}+s_{2,1})^2 + (s_{1,2}+s_{2,2})^2\ \right]$$

and

$$\begin{aligned} U_P^1 \quad = \quad & p_1\,\frac{s_{1,1}+s_{2,1}}{k} - U_A^0 + h_2 + \left[\frac{1}{2k} - \frac{\alpha\, v}{2}\right]\ \left[\ (s_{1,1}+s_{2,1})^2 + (s_{1,2}+s_{2,2})^2\ \right] \\ & - \frac{1}{k}\ \left[\ s_{1,1}\,(s_{1,1}+s_{2,1}) + s_{1,2}\,(s_{1,2}+s_{2,2})\ \right] \end{aligned}$$

To maximize this, FOCs for $s_{1,1}$ and $s_{1,2}$ are

$$\begin{aligned}
\frac{1}{k}\,p_1 + \left[\frac{1}{2k} - \frac{\alpha\,v}{2}\right]\;2\,(s_{1,1}+s_{2,1}) - \frac{1}{k}\;(2\,s_{1,1}+s_{2,1}) &= 0 \\
\left[\frac{1}{2k} - \frac{\alpha\,v}{2}\right]\;2\,(s_{1,2}+s_{2,2}) - \frac{1}{k}\;(2\,s_{1,2}+s_{2,2}) &= 0
\end{aligned}$$

or

$$\begin{aligned}
p_1 - (1+\alpha\,v\,k)\;s_{1,1} - \alpha\,v\,k\;s_{2,1} &= 0 \\
-(1+\alpha\,v\,k)\;s_{1,2} - \alpha\,v\,k\;s_{2,2} &= 0
\end{aligned}$$

These implicitly define principal 1's "best response" or "reaction functions".

Similarly, for principal 2, we get the conditions

$$\begin{aligned}
p_2 - (1+\alpha\,v\,k)\;s_{2,2} - \alpha\,v\,k\;s_{1,2} &= 0 \\
-(1+\alpha\,v\,k)\;s_{2,1} - \alpha\,v\,k\;s_{1,1} &= 0
\end{aligned}$$

defining his best response or reaction functions.

Solve all four together for Nash equilibrium. Actually simpler: Principal 1's first and 2's second equation solve for $s_{1,1}$ and $s_{2,1}$:

$$s_{1,1} = \frac{1 + \alpha\, v\, k}{1 + 2\,\alpha\, v\, k}\; p_1, \qquad s_{2,1} = -\frac{\alpha\, v\, k}{1 + 2\,\alpha\, v\, k}\; p_1$$

Add to get aggregate bonus coefficient for the first outcome:

$$s_1 = s_{1,1} + s_{2,1} = \frac{1}{1 + 2\,\alpha\, v\, k}\; p_1$$

Similarly for second output.

Result: Compared to cooperative or joint maximization situation,
each principal offers stronger bonus coefficient for "own" output

$$\frac{1 + \alpha\, v\, k}{1 + 2\, \alpha\, v\, k} > \frac{1}{1 + \alpha\, v\, k}$$

But each offers negative coefficient on the "other's" output, and total

$$\frac{1}{1 + 2\, \alpha\, v\, k} < \frac{1}{1 + \alpha\, v\, k}$$

is smaller in Nash than in joint.

If n principals, change 2 in denominator to n.
More powerful incentives if each principal forbidden to observe
or condition on the other's outcome.

Brief intuitive account of multidimensional issues

Multiple periods

Career concerns motivate young workers, need less cash payment.
"Efficiency wage" in ongoing relationship can deter cheating.
Average outcome over time can be more accurate indicator of effort.
Early revelation of skill harder – "ratchet effect".

Multiple tasks

Incentives can be usefully conditioned on outcomes if they
are informative about effort, regardless of direct value to principal.
Different degrees of verifiability (errors in observation etc.)
create problems of "paying for A while hoping for B".
Substitute tasks make incentive problem harder; complements, easier
Implications for splitting up tasks among different agencies.

Multiple agents

Relative performance can be more precise measure of
relative effort or type if the random errors are
highly positively correlated across agents.
Can use this by setting up competition – market or auction.
But such schemes are vulnerable to collusion among agents.

Multiple types of agents

If agent internalizes principal's objective, problem becomes easier.
Principal can screen for such agents -
charities attract dedicated workers at low wages.
But beware of multidimensionality of types - may attract low-skilled.
Extrinsic material incentives can lower / destroy
intrinsic personal / social ones.