ECO 352 – International Trade – Spring Term 2010
Week 5 Precepts – March 1
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Here we give algebraic derivations of these theorems.

Assumptions and notation

Two goods, X and Y, are produced using two factors, K and L. The factors are mobile across sectors. Denoting their allocations to the two uses by K_X etc., we have

$$K_X + K_Y = K, \qquad L_X + L_Y = L\,. \tag{1}$$

The production functions in the two sectors are

$$X = F_X(K_X, L_X), \qquad Y = F_Y(K_Y, L_Y)\,.$$

Both are assumed to have constant returns to scale. Let A_{KX} denote the amount of capital that goes into the production of one unit of X, and similarly for the other factor and good. In class we gave numerical examples where fixed coefficients ruled in production. Now we allow substitution.

Let P_X and P_Y denote the prices of the two goods, and R and W denote the rate of return to capital and the wage rate respectively.

Cost-minimization

The input coefficients (A_{KX}, K_{LX}) are chosen on the production isoquant that corresponds to 1 unit of the output of X:

$$F_X(A_{KX}, A_{LX}) = 1$$

to minimize the cost

$$R\ A_{KX} + W\ A_{LX}\,.$$

The condition for this is that the factor price ratio equals the marginal rate of technical substitution along the isoquant:

$$\frac{R}{W} = -\left.\frac{dA_{LX}}{dA_{KX}}\right|_{F_X(A_{KX},A_{LX})=1}\,. \tag{2}$$

Figure 1 shows the usual tangency condition. It shows the slope dA_{LX}/dA_{KX} for very small changes dA_{KX} and dA_{LX}. Then the equality of the slope and the factor price ratio holds to a sufficiently good approximation (in calculus terms, to the first order of smallness). We will use this later. A similar condition holds for the input coefficients in Y.

The cost-minimizing input coefficients A_{ij} are thus all functions of the factor price ratio (R/W). We will assume that the X good is relatively more K-intensive than the Y-good. Therefore for any given (R/W), we have

$$\frac{A_{KX}}{A_{LX}} > \frac{A_{KY}}{A_{LY}}\,, \qquad \text{or} \qquad \frac{A_{KX}}{A_{KY}} > \frac{A_{LX}}{A_{LY}}\,, \tag{3}$$

or equivalently

$$A_{KX}\ A_{LY} - A_{LX}\ A_{KY} > 0\,.$$

For equilibrium in the output markets, we have the zero-profit (price equals unit cost) conditions:

$$P_X = R\ A_{KX} + W\ A_{LX}\,, \qquad P_Y = R\ A_{KY} + W\ A_{LY}\,. \tag{4}$$

Stolper-Samuelson

Now let P_X and P_Y change. We calculate the effect of very small changes; then the effect of larger changes can be found by adding up the small ones.

The total differential of the X-sector price-cost equation in (4) is

$$dP_X = dR\ A_{KX} + R\ dA_{KX} + dW\ A_{KX} + W\ dA_{LX}\,.$$

But (2) gives

$$R\ dA_{KX} + W\ dA_{LX} = 0$$

to our first-order approximation. Therefore

$$dP_X = dR\ A_{KX} + dW\ A_{KX}\,. \tag{5}$$

This is more easily stated and interpreted in terms of proportional changes like dP_X/P_X. Therefore write it as

$$\frac{dP_X}{P_X} = \frac{R\,A_{KX}}{P_X}\ \frac{dR}{R} + \frac{W\,A_{LX}}{P_X}\ \frac{dW}{W}\,.$$

Write $\theta_{KX} = R\ A_{KX}/P_X$; this is the share of the cost of capital in the unit cost of good X. Similarly for both goods and both factors. Then (5) can be written as

$$\frac{dP_X}{P_X} = \theta_{KX}\ \frac{dR}{R} + \theta_{LX}\ \frac{dW}{W}\,.$$

Similarly for good Y,

$$\frac{dP_Y}{P_Y} = \theta_{KY}\ \frac{dR}{R} + \theta_{LY}\ \frac{dW}{W}\,.$$

We can solve these as a pair of linear equations for the factor price changes in terms of the given output price changes. We have

$$\frac{dR}{R} = \frac{1}{\Delta}\left[\theta_{LY}\ \frac{dP_X}{P_X} - \theta_{LX}\ \frac{dP_Y}{P_Y}\right] \tag{6}$$

$$\frac{dW}{W} = \frac{-1}{\Delta}\left[\theta_{KY}\ \frac{dP_X}{P_X} - \theta_{KX}\ \frac{dP_Y}{P_Y}\right] \tag{7}$$

where Δ is the determinant of the coefficient matrix

$$\Delta = \theta_{KX}\ \theta_{LY} - \theta_{KY}\ \theta_{LX}\,.$$

But in each sector the factor cost shares must add up to one:

$$\theta_{KX} + \theta_{LX} = 1, \qquad \theta_{KY} + \theta_{LY} = 1\,.$$

Therefore

$$\begin{aligned} \Delta &= \theta_{KX}\,(1-\theta_{KY}) - \theta_{KY}\,(1-\theta_{KX}) \\ &= \theta_{KX} - \theta_{KY} && (8) \\ &= (1-\theta_{LX}) - (1-\theta_{LY}) \\ &= \theta_{LY} - \theta_{LX} && (9) \end{aligned}$$

The sign of Δ relates to our relative factor intensity assumption.

$$\theta_{KX} > \theta_{KY} \quad \text{if and only if} \quad \frac{\theta_{KX}}{1-\theta_{KX}} > \frac{\theta_{KY}}{1-\theta KY}\,,$$

$$\text{or} \quad \frac{R\ A_{KX}/P_X}{W\ A_{LX}/P_X} > \frac{R\ A_{KY}/P_Y}{W\ A_{LY}/P_Y}\,, \quad \text{or} \quad \frac{A_{KX}}{A_{LX}} > \frac{A_{KY}}{A_{LY}}\,,$$

which we are assuming in (3). Therefore $\Delta > 0$.

Now we are ready to interpret the effect of a change in P_X on R and W from (6) and (7). Holding P_Y constant and using the expression in (9) for Δ, we have

$$\frac{dR}{R} = \frac{\theta_{LY}}{\theta_{LY} - \theta_{LX}}\ \frac{dP_X}{P_X}\,.$$

But $\theta_{LY}/(\theta_{LY} - \theta_{LX}) > 1$; therefore a 1% increase in P_X leads to an increase of more than 1% in R, the price of the factor used more intensively in production of X. And

$$\frac{dW}{W} = \frac{-\theta_{KY}}{\theta_{KX} - \theta_{KY}}\ \frac{dP_X}{P_X}\,,$$

so an increase in P_X leads to a fall in W, the price of the factor used less intensively in production of X.

The effects of a change in P_Y holding P_X unchanged can be calculated similarly; this is left as an exercise. You can also look at relative prices. Measure all prices relative to P_Y, and find the effects of a change in P_X/P_Y on R/P_Y and W/P_Y. Remember that

$$\frac{d(P_X/P_Y)}{P_X/P_Y} = \frac{dP_X}{P_X} - \frac{dP_Y}{P_Y} \text{ etc.}$$

Rybczynski

Here we look at the effects of changes in K and L on X and Y, holding P_X, P_Y constant. The latter implies that R and W are also unchanged ($dP_X = dP_Y = 0$ implies $dR = dW = 0$ by (6) and (7)). Then all four input coefficients A_{KX} etc. are also unchanged, and we can write the factor market clearing equations (1) as

$$A_{KX}\, X + A_{KY}\, Y = K, \qquad A_{LX}\, X + A_{LY}\, Y = L\,. \tag{10}$$

Write $\lambda_{KX} = A_{KX}\, X/K$, this is the proportion of the total available capital that is employed in X-production. Similarly for the other factor and good. Then, doing algebra similar to that for the cost fractions θs, the condition that X-production is relatively K-intensive can be written as

$$\lambda_{KX} > \lambda_{LX}, \quad \text{or} \quad \lambda_{LY} > \lambda_{KY}\,.$$

Now differentiate (10) totally and solve for the output changes. The algebra is very similar to that of the factor price solutions and is left as an exercise. The result of a change in K is

$$\frac{dX}{X} = \frac{\lambda_{LY}}{\lambda_{LY} - \lambda_{KY}}\;\frac{dK}{K}\,, \qquad \frac{dY}{Y} = \frac{-\lambda_{LX}}{\lambda_{KX} - \lambda_{LX}}\;\frac{dK}{K}\,.$$

Therefore a 1% increase in K increases the output of X (the good that uses K relatively more intensively) by more than 1%, and decreases the output of the other good.

To reinforce your understanding of this material, read the Mathematical Postscript to Chapter 4, pp. 666–669 in K-O, for this derivation using a somewhat different notation.