ECO 199 – GAMES OF STRATEGY
Spring Term 2004
PROBLEM SET 2 – ANSWER KEY

The distribution of grades was as follows.

Range	Numbers
100-	5
90-99	26
80-89	9
70-79	7
0-69	3

Again a generally good performance, but not quite as good as on the first problem set.

Question 1:

COMMON ERRORS: [1] Most of the mistakes were in part (b), where students seem to be uneasy with a two-variable problem. In order to get back to a one-variable problem, they have either stated that the two profits have to be equal or randomly plugged in one of the two best-response (-8). Unlike the pricing problem on pp. 124-8 of the book, but like the quantity problem on pp. 147-9 or the ice cream pricing problem done in class, the situation is not symmetric between the firms so their prices and profits need not be equal. [2] There were pure computation mistakes in (a) and (b) (half the points taken off). [3] Among good problem sets, a common mistake in (c) has been to say that joint-profit maximization cannot be a NE because one of the two firms gets less than at the NE (-2). [4] Finally, one bonus point was given to students who noticed that collusion is good for both consumers and sellers in the case of complements.

(a) (6 points for the algebra of each best response function, 4 for the graph, and 4 for solving for the Nash equilibrium)

La Boulangerie's profit:

$Y_1 = P_1 Q_1 - Q_1 = P_1 (10 - P_1 - 0.5 P_2) - (10 - P_1 - 0.5 P_2) = - (P_1)^2 + 11 P_1 - 0.5 P_1 P_2 + 0.5 P_2 - 10$

To choose P_1 to maximize this, regarding P_2 as fixed,

set $dY_1/dP_1 = - 2 P_1 + 11 - 0.5 P_2 = 0$ and solve for P_1 to get $P_1 = 5.5 - 0.25 P_2$.

This is La Boulangerie's best response function.

Alternatively, follow the method of completing the square as on p. 125 of the text. Rearranges the expression for profit as

$Y_1 = - (0.5 [11 - 0.5 P_2] - P_1)^2 - (0.5 [11 - 0.5 P_2])^2 - 10 + 0.5 P_2$.

The only term with P_1 here is a squared term that enters negatively. To make Y_1 as large as possible we must make this term as small as possible, which we do by setting the terms inside the parentheses equal to zero. This gives: $0.5 [11 - 0.5 P_2] = P_1$ or $P_1 = 5.5 - 0.25 P_2$.

La Fromagerie's profit:

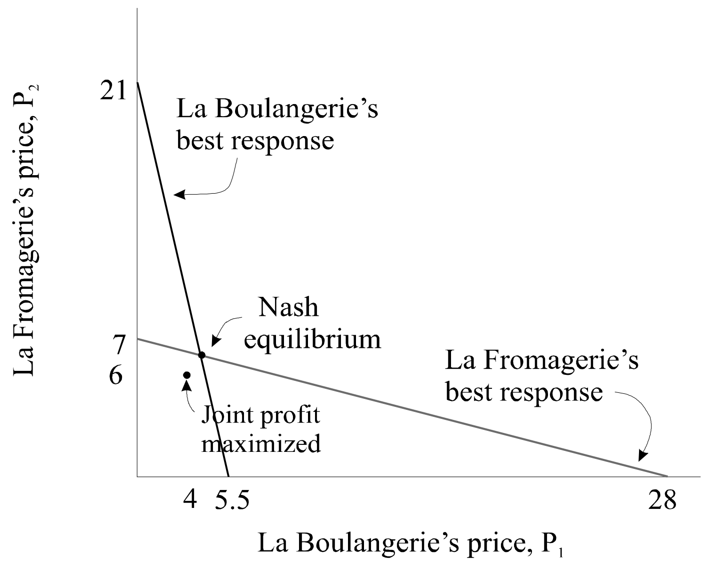
$Y_2 = P_2 Q_2 - 2 Q_2 = P_2 (12 - 0.5 P_1 - P_2) - 2 (12 - 0.5 P_1 - P_2) = - (P_2)^2 + 14 P_1 - 0.5 P_1 P_2 + P_1 - 24$.

Setting $dY_2/dP_2 = - 2 P_2 + 14 - 0.5 P_1 = 0$ and solving for P_2 gives $P_2 = 7 - 0.25 P_1$. This is La Fromagerie's best response function. You can get the same answer by rearranging:

$Y_2 = - (0.5 [14 - 0.5 P_1] - P_2)^2 - (0.5 [14 - 0.5 P_1])^2 - 24 + P_1$.

Make the squared term containing P_2 as small as possible with: $0.5 [14 - 0.5 P_1] - P_2 = 0$ or $P_2 = 7 - 0.25 P_1$.

Graphing best response functions shows the Nash equilibrium at the intersection of the two curves (lines). To find the solution for the equilibrium prices analytically, you substitute one best response function into the other; substitute La Fromagerie's best response function for P_2 into La Boulangerie's best response function. This process yields: $P_1 = 5.5 - 0.25\ (7 - 0.25\ P_1)$ or $P_1 = 4$. Given this value for P_1, you can find $P_2 = 7 - 0.25\ (4) = 6$.



(b) (8 points) Colluding to set prices to maximize the sum of profits means the firms maximize the joint profit function: $Y = Y_1 + Y_2 = 12\ P_1 + 14.5\ P_2 - (P_1)^2 - (P_2)^2 - P_1\ P_2 - 34$. Since the same Y is to be maximized with respect to both P_1 and P_2, we must differentiate it with respect to each holding the other constant. This gives two equations: with respect to P_1 we have $12 - 2\ P_1 - P_2 = 0$ and with respect to P_2 we have $14.5 - 2\ P_2 - P_1 = 0$. Solving these two equations jointly for P_1 and P_2 we get $P_1 = 3.17$ and $P_2 = 5.67$.

To answer without calculus, realize that the firms are individually choosing P_1 and P_2, but both firms have the same goal of maximizing Y (rather than Y_1 or Y_2). Then best response curves can be found by completing the squares on Y (as above) with respect to P_1 and P_2 separately. This yields two expressions for Y in which the relevant price appears only in a squared term that enters negatively in the profit expression. Setting the squared term equal to zero in each case yields two equations in P_1 and P_2: $6 - 0.5\ P_2 = P_1$ (when completing the square with respect to P_1) and $7.25 - 0.5\ P_1 = P_2$ (when completing the square with respect to P_2). Solving these two equations simultaneously yields the solution $P_1 = 3.17$ and $P_2 = 5.67$.

(c) (4 points) When firms choose to maximize joint profit, they act as a single firm and ignore any individual incentives that they might have to deviate from the joint profit goal. However, plugging the joint-profit-maximizing value of La Boulangerie's price into La Fromagerie's individual best-response rule will not yield La Fromagerie's joint-profit-maximizing price (and vice versa). If La Boulangerie charges the joint profit-maximizing price $P_1 = 3.17$, then La Fromagerie's best response is $P_2 = 7 - 0.25 * 3.17 = 7 - 0.79 = 6.21$, which exceeds the joint-profit-maximizing $P_2 = 5.67$. This is because the higher price benefits La Fromagerie's profits (while reducing the demand for bread and therefore La Boulangerie's profits). Thus the joint-profit-maximizing prices are not best responses to each other and cannot be a Nash equilibrium.

(d) (3 points) When firms produce substitutes, a drop in price at one store hurts the sales of the other; thus, as your rival drops her price, you want to drop yours also to attempt to maintain sales (and profits). In the bistro example, this result led to best response curves that were positively sloped and Nash equilibrium prices that were lower than the joint profit maximizing prices. Here, the firms produce complements so a drop in the price at one store leads to an increase in sales at the other; in this case, as one store drops its price, the other can safely increase its price somewhat and still maintain sales (and profits). Thus, the best response curves are negatively sloped in this example and the Nash equilibrium prices are higher than the joint profit maximizing prices. Thus in the case of complements, letting the firms collude yields higher profit for the firms jointly and lower prices for the consumers, a rare example of a change that is good for everyone. (One way to interpret the Microsoft-Netscape interaction is that Microsoft claims operating systems and browsers are complements so letting Microsoft monopolize both would be in everyone's interests, whereas the Department of Justice and Netscape claim that operating systems and browsers are substitutes - because browsers can now perform many of the functions that were once the preserve of operating systems, such as starting a program and printing a document - so a Microsoft monopoly over both would hurt consumers.)

Question 2:

COMMON ERRORS: There were very few mistakes in solving the game. They range from failure to find the NE in the simultaneous game (with the right pay-offs), to wrong computation of the expected pay-offs and to total failure in formalizing the game as a sequence of simultaneous games (up to 20 points taken off). No points were taken off if students have described the equilibrium path instead of the full equilibrium strategies. In Exercice 10, the vast majority of the students have got full credit (even those who had made mistakes in 9). The patent race is the most common example, but the movie industry is not far behind. The most exotic one (based on personal experience?) is: "two firms competing for the one and only grad student in Turkish; the two firms can either wait and receive more and more transcripts or hire him right away."

Exercise 9: (3 points for the first step, 5 points each for each of the other 4 steps)

As usual for sequential-move games, we begin at the end. Suppose both aristocrats have walked 5 steps and are right next to each other, but neither has yet fired. The payoffs are shown below, and it is easy to see that "Shoot" is the dominant strategy at this step for each player; the resulting payoffs are (0,0).

		Chagrin	
		Shoot	Not
Renard	Shoot	0, 0	1, -1
	Not	-1, 1	0, 0

Knowing the equilibrium at the 5-step subgame, we use rollback to consider the payoffs after each player has walked 4 steps:

		Chagrin	
		Shoot	Not
Renard	Shoot	0, 0	0.6, -0.6
	Not	-0.6, 0.6	0, 0

Payoffs above are derived as follows: [1] top right cell: if Renard shoots and Chagrin does not, then Renard hits with probability .8 and gets 1, and misses with probability .2, in which case Chagrin has a sure shot after 5 moves so Renard gets -1. Renard's expected payoff: $0.8 \times 1 + 0.2 \times (-1) = 0.6$. Chagrin's expected payoff: $0.8 \times (-1) + 0.2 \times 1 = -0.6$. Bottom left cell calculations are similar. [2] top left cell: if both shoot, Chagrin gets $0.8 \times 0.8 \times 0 + 0.8 \times 0.2 \times 1 + 0.2 \times 0.8 \times (-1) + 0.2 \times 0.2 \times 0 = 0$. [3] bottom right cell: if neither shoots on step 4, the game goes to step 5, where we know equilibrium payoff is (0,0). Thus, we make use of the analysis of Step 5 performed above.

Now analyzing the payoff table for step 4, we see “Shoot” is the dominant strategy for both, and the resulting payoff is (0,0).

For Step 3, the duelists are six steps apart and the probability of hitting if you shoot is 0.6. Similar calculations as those for Step 4 (for instance, in top right and bottom left cells, use: $0.6 \times 1 + 0.4 \times (-1) = 0.2$) lead to the following payoff table:

		Chagrin	
		Shoot	Not
Renard	Shoot	0, 0	0.2, -0.2
	Not	-0.2, 0.2	0, 0

Again, “Shoot” is dominant for each player, and yields the payoffs (0,0).

Next move back to step 2, with the pair 8 paces apart and the probability of hitting down to 0.4. The payoff table is now:

		Chagrin	
		Shoot	Not
Renard	Shoot	0, 0	-0.2, 0.2
	Not	0.2, -0.2	0, 0

Things have changed: “Not Shoot” is now the dominant strategy for both. The resulting equilibrium payoff is still (0,0).

Finally, at step 1, with the pair in their starting positions and the probability of hitting only 0.2, the payoff matrix is as shown below:

		Chagrin	
		Shoot	Not
Renard	Shoot	0, 0	-0.6, 0.6
	Not	0.6, -0.6	0, 0

“Not shoot” is again the dominant strategy with payoffs of (0, 0).

You can think of this as one way to formalize and examine rigorously the idea “Don’t shoot till you see the whites of their eyes” !

(4 points) Actual play of the game will result in both players shooting on step 3. The equilibrium strategy (complete plan of action) for either player is: “Do not shoot on steps 1 and 2 no matter what. At any step, if the other player has shot and missed, while you have yet to shoot, then wait until step 5 to shoot. If you arrive at step 3 (or later) and the other player has not yet shot, then shoot at once.”

Exercise 10: (3 points)

Consider two firms who are trying to decide how much time to spend developing their product before releasing it into the market. The longer a firm waits, the better the product it will be selling, and (other things equal) the better its chance of taking over the market. However, the longer one firm waits, the greater the chance that the other firm will release its product first; if that product is successful, the firm that delayed may be shut out of the market. Thus, each firm has to choose between waiting to "shoot" (thus increasing its chance of hitting the "bullseye") and "shooting" early (in order to be sure of getting off a shot before the other player hits the target).

Question 3:

COMMON ERRORS: Very few mistakes. A few students have got zero because they agreed with Charlie. Even without knowing any game theory, you should have guessed from the names that Charlie could never be right; Lucy will always fool him and get the better deal! That is why I used these names. By the way, the problem itself was posed by Marilyn vos Savant in one of her columns a few years ago.

(a) (3 points) Charlie's mistake is to think that heads and tails are equally likely. This is not a coin toss; each player is consciously choosing whether to show heads and tails. (OK to do the mixed strategy equilibrium calculation first and then answer the "Why not?" question.)

(b) (12 points) In fact one must calculate the probabilities in the equilibrium mixture. Here is the payoff matrix showing Lucy's payoffs:

		Charlie	
		Heads	Tails
Lucy	Heads	- 3	2
	Tails	2	- 1

Lucy's mix of (Heads:p, Tails:1-p) gives her $-3p + 2(1-p) = 2 - 5p$ if Charlie chooses Heads, and $2p - (1-p) = 3p - 1$ if Charlie chooses Tails. Her optimal choice, which leaves Charlie no opportunity to exploit her systematically, is given by $2 - 5p = 3p - 1$, or $p = 3/8$. Her expected payoff is then $2 - 5 * 3/8 = 1/8$.

When Lucy mixes in these proportions, her expected payoff is
$-3 * 3/8 + 3 * 5/8 = 1/8$ if Charlie chooses pure Heads
$2 * 3/8 - 1 * 5/8 = 1/8$ if Charlie chooses pure Tails
and therefore also 1/8 for any q:(1-q) mixture Charlie might choose. In other words, by choosing her equilibrium mix, she gets the same expected payoff no matter what Charlie chooses. This is exactly the idea that in a zero-sum game, by choosing your correct mixture you can ensure that the other player is unable to exploit your choice to his advantage.

Question 4:

COMMON ERRORS: This question proved to be the most difficult one. Very few students were willing to say that there is no NE in (b). Instead, most of them have discussed weakly dominant strategies. Full credit has been given if they have explicitly mentioned that they consider weakly dominant strategies (-2 otherwise), if they have considered all possible cases (-1) and if all questions are consistent (from -3 to -5). A few students seem to have problems with the link between conditions in parts (a) and (b)-©) (how "not or" becomes "and"). Finally, one bonus point to students who checked that p is indeed between 0 and 1.

(a) (4 points) Row has a dominant strategy if $C > A$; Column has a dominant strategy if $C > B$.

(b) (5 points) If neither of the conditions in (a) hold, then we have $C < A$ and $C < B$. In this case there can be no Nash equilibrium in pure strategies.

(c) (5 points) Under the same conditions as for part b, namely $A > C$ and $B > C$, there is no Nash equilibrium in pure strategies but there will be an equilibrium in mixed strategies.

Additional information: This is a special property of two-person zero-sum games where each player has exactly two pure strategies: either they are dominance solvable, or their Nash equilibrium must be in mixed strategies.

(d) (6 points) The mixed strategy equilibrium value of p (probability of choosing Up) for Row must satisfy $pA = (B-C)(1-p)$ or $p/(1-p) = (B-C)/A$. Then $p = (B-C)/(A+B-C)$.

Additional information: Note that $B > C$ (and $A > 0$) ensures $p < 1$.

The Column player's mix can be found similarly.