ECO 305 – Fall 2003
Microeconomic Theory – A Mathematical Approach
Final Examination – Draft Answer Key

Review the answer key carefully; this should resolve most of your questions about your grade. Note that we are grading on a full scale of 0–100, and that assignment of credit for partially correct math problems often requires judgement which must be exercised by comparing performance across students, not just by reading your own exam. If you are not satisfied and want regrading, return your whole exam, with a brief statement of your issue, in an envelope to Professor Dixit's mailbox by 5 p.m. on Tuesday February 3. We will regrade the entire exam, and you may lose points instead of gaining any.

Here is some statistical information. Median 67, Mean 67.5, Standard deviation 15.2.

Distribution

Range	Number
90–99	8
80–89	15
70–79	15
60–69	20
50–59	14
40–49	9
< 40	3

Average by question

Question	Average
1	13.1
2	12.6
3	10.3
4	8.7
5	11.0
6	19.7
7	23.6

Question 1: (15 points, 3 each)

Definitions are judged by their completeness, clarity, and succinctness. Simple figures in (b), (c) and (d) will help. Small lapses of wording that do not cause errors or confusion in reading are OK; small but significant errors cost 1 point; more substantial errors 2 or more points. This is the grader's judgement call, and like balls and strikes, cannot be questioned.

COMMON ERRORS: These were mostly done very well, with occasional lapses of knowledge and/or memory. A few people thought Pareto efficiency was just what it is not – a situation where one person could be made better off without making any one else worse off. They confused the core with the locus of all Pareto efficient points. Some people thought that Nash equilibrium involved agents maximizing their payoffs regardless of the other agents' strategies, instead of taking the other players' strategies as given, perhaps because they had the prisoner's dilemma game in mind. Occasionally people got confused about when a constraint is binding and when it is slack while talking about complementary slackness. While discussing the utility function many people didn't realize that tastes are independent of prices and income and that it is only the indirect utility function which contains these as arguments. Some people confused the expected value with the expected utility. While dealing with dead weight loss to monopoly some people confused this with dead weight loss in case of taxation and others did not explain why monopolies set higher prices and produce lower quantities though in other respects the answers were correct. The answers below say where you should have learned or found the concepts.

(a) Principle of complementary slackness

Lecture Overheads (LO) Sep. 18 p. 2, also used in Precepts Weeks 4, 11.

In the solution to an optimization problem with weak inequality ($\geq$ or $\leq$) constraints, for each constraint at least one of the following must be true: (i) the constraint is binding (holds as $=$), (ii) its Lagrange multiplier equals zero.

(b) Utility function

Binger-Hoffman (BH) pp. 133; LO Sep. 23 p. 1.

A utility function assigns a number to each consumption bundle such that the relative ordinal numerical ranking of any two consumption bundles corresponds to their relative ranking in the consumer's preferences.

(c) Pareto efficiency (also called Pareto optimality)

BH p. 239, LO Nov. 4, General Equilibrium and Efficiency handout p. 5.

An allocation is Pareto optimal or Pareto efficient if it is not possible to find an another allocation that makes one person better off without making some other person worse off.

(d) Dead-weight loss from monopoly

BH pp. 416; LO Nov. 11 p. 2.

A monopolist charges a price higher than marginal cost and sells a quantity smaller than what would equate price and marginal cost. The resulting loss of consumer surplus exceeds the increase in the monopolist's profit. The difference, which is economic surplus lost, is called the dead-weight loss from monopoly.

(e) Nash Equilibrium

BH p. 451, LO Nov. 13 p. 1, Oligopoly handout p. 2.

A Nash equilibrium of a game is a list of strategies, one for each player, such that the listed strategy of each maximizes his payoff given the listed strategies of the others.

(f) Expected utility

BH p. 534, LO Nov. 20, p. 2.

Expected utility is a representation of preferences under uncertainty by the expected value (probability-weighted average) of the list of (von Neumann-Morgenstern) utilities in each possible outcome of the random prospect (scenario).

(g) The winner's curse

LO Dec. 11, p. 1.

This arises in common value auctions where each player gets an estimate of the value. Even though each player's estimate may be unbiased, the winner is selected to be the one with the highest bid and therefore (usually) the one with the highest estimate, so the expected value of his estimate exceeds the true value.

Question 2: (15 points, 5 each)

As with the definitions, the criteria for grading are completness, accuracy, and succinctness.

COMMON ERRORS: Again, mostly a good job, but some confusion about the basic concepts. Some were confused about what "strategic commitment" was. Many also thought that sunk costs need to be covered in the short run for the firm to continue operating (rather than non-sunk costs). Regarding idiosyncratic risk, some people gave an answer about individuals being potentially risk loving, risk averse or risk neutral. They didn't realize what the question referred to (distinction from systematic or undiversifiable risk). Again the answers cite references for these concepts.

(a) Lagrange multiplier

BH pp. 73-5, lO Sep. 16 p. 5 and numerous repetitions.

The Lagrange multiplier in a constrained optimization problem equals the marginal increase in the optimum value of the objective function that is made possible if the constraint can be relaxed slightly. In economic terms, it is the shadow value or shadow price of the constraint.

(b) Quasi-linear preferences

LO Oct. 7, p. 1, Oct. 23, p. 4 and later use in Precept 8 and oligopoly theory.

If preferences are quasilinear, so long as income is sufficient that the consumer wants to choose a positive quantity of the good in which utility is linear, any marginal increase in income is spent entirely on this good; the income effect on other goods (the ones in which utility is nonlinear) is zero. In economic theory, this helps isolate out those goods from the rest of the economy for easier partial equilibrium analysis.

(c) Sunk costs

BH pp. 325-6, LO Oct. 14 p. 2, handout on Summary of (Short Run) Cost Concepts.

Sunk costs arise in the short run and they cannot be avoided by shutting down production; they are typically contractual commitments like debt servicing that the firm must fulfill even if it produces nothing. Therefore in the short run, it is optimal for a firm to produce even though it makes an accounting loss so long as this loss does not exceed the sunk costs, that is, so long as its avoidable (non-sunk) costs are covered.

(d) Strategic commitment

Precept Week 9, LO Nov. 18 pp. 3-4.

In a two-stage game, this is the ability to make a move in the first period that changes the outcome of the second stage in one's favor. In economics it can be used by incumbent firms to prevent the entry of new firms so they can go on enjoying the profits of a monopoly or a cartel. They will use this so long as the extra profit exceeds the cost of making the commitment.

(e) Individual or idiosyncratic risk

LO Nov. 25 p. 4, Dec. 2, p. 4, Financial Markets handout pp. 7, 13.

This is risk that is uncorrelated with the risk in the market as a whole, or with the aggregate wealth of the economy. Individual or idiosyncratic risk can be fully traded away or diversified in a competitive financial market; therefore no risk premium is is given for bearing such risk.

Question 3: (20 points)

COMMON ERRORS: Major – Forgetting the definitions in (a) and (b), which then leads to errors in (c) and (d). Minor – Failing to define notation. In between – saying that X_i^H is the derivative of *income* $\partial I/\partial p_i$. No; it is the derivative of the *expenditure function.*

WHERE WAS THIS MATERIAL COVERED: For parts (a)-(c), see the Lecture Handout of Sep. 25, pp. 3, 4, 6.

(a) (4 points) Shepherd's lemma: If the expenditure function is

$$M^*(P_1, P_2, \ldots P_n, u) = \min \{ P_1 X_1 + \ldots + P_n X_n \mid U(X_1, \ldots X_n) \geq u \},$$

then

$$X_i^H = \frac{\partial M^*}{\partial P_i}$$

(b) (4 points) Slutsky equation

$$\frac{\partial X_i^H}{\partial P_j} = \frac{\partial X_i^M}{\partial P_j} + X_j^M \frac{\partial X_i^M}{\partial I}$$

(c) (5 points) Using Shepherd's lemma,

$$\frac{\partial X_i^H}{\partial P_j} = \frac{\partial}{\partial P_j}\left(\frac{\partial M^*}{\partial P_i}\right) = \frac{\partial}{\partial P_i}\left(\frac{\partial M^*}{\partial P_j}\right) = \frac{\partial X_j^H}{\partial P_i}$$

(d) (7 points) Combining (b) and (c) above, we see that

$$\frac{\partial X_i^M}{\partial P_j} = \frac{\partial X_j^M}{\partial P_i}$$

will be true if and only if

$$X_j^M \, \frac{\partial X_i^M}{\partial I} = X_i^M \, \frac{\partial X_j^M}{\partial I}$$

Since the quantities are both positive, we can divide by $X_i^M \, X_j^M$ and multiply by I to write this as

$$\frac{I}{X_i^M} \, \frac{\partial X_i^M}{\partial I} = \frac{I}{X_j^M} \, \frac{\partial X_j^M}{\partial I}$$

Question 4: (20 points)

COMMON ERRORS: Major: (1) In part (d), doing a few first steps and then triumphantly proclaiming the given answer, when at best some intermediate steps were omitted, and at worst the calculation earlier had errors and would never have led to the correct answer. Did these people seriously think we would just look at the last line and give them full credit? (2) Using wrong concepts of MRTS, elasticity of substitution etc. Minor: (1) Failing to say that the scaling factor in (a) has to be > 1. (2) Algebra errors; these made early on could of course turn into major errors later.

WHERE WAS THIS MATERIAL COVERED: See LO Oct. 9, pp. 3-4. We did similar problems in Precepts Week 3 Question 3, and Problem Set 3 Question 1. See also BH pp. 209, 267.

(a) (3 points) If (K, L) is changed to $(\theta K, \theta L)$ with $\theta > 1$, then output changes to

$$[\,(\theta\, K)^{1/3} + 2\,(\theta\, L)^{1/3}\,]^4 = \theta^{4/3}\;[\,K^{1/3} + 2\,L^{1/3}\,]^4 > \theta\;[\,K^{1/3} + 2\,L^{1/3}\,]^4$$

Therefore the function has increasing returns to scale.

(b) (4 points) The marginal products are

$$\begin{aligned}\frac{\partial Q}{\partial K} &= 4\;[\,K^{1/3} + 2\,L^{1/3}\,]^3\;\frac{1}{3}\;K^{-2/3}\\ \frac{\partial Q}{\partial L} &= 4\;[\,K^{1/3} + 2\,L^{1/3}\,]^3\;2\;\frac{1}{3}\;L^{-2/3}\end{aligned}$$

Therefore the marginal rate of input substitution is

$$-\left.\frac{\partial K}{\partial L}\right|_{Q \text{ constant}} = \frac{\partial Q/\partial L}{\partial Q/\partial K} = 2\;\left(\frac{L}{K}\right)^{-2/3}$$

Therefore as L increases and K decreases along an isoquant, the marginal rate of substitution decreases.

(c) (5 points) Since the production function has a diminishing MRS, the tangency cost minimization condition is necessary and sufficient (alternatively you could do Lagrange):

$$\frac{w}{r} = -\left.\frac{\partial K}{\partial L}\right|_{Q \text{ constant}} = 2\;\left(\frac{L}{K}\right)^{-2/3}$$

Then

$$\frac{L}{K} = \left(\frac{2\,r}{w}\right)^{3/2}$$

So the elasticity of substitution is 3/2.

(d) (8 points) Substitute for L in terms of K in the production function:

$$\begin{aligned} Q &= \left[K^{1/3} + 2\, K^{1/3} \left(\frac{2\,r}{w} \right)^{1/2} \right]^4 \\ &= K^{4/3} \left[1 + 2^{3/2}\, \frac{r^{1/2}}{w^{1/2}} \right]^4 \\ &= K^{4/3}\, r^2 \left[r^{-1/2} + 2^{3/2}\, w^{-1/2} \right]^4 \end{aligned}$$

Therefore

$$K = Q^{3/4}\, r^{-3/2} \left[r^{-1/2} + 2^{3/2}\, w^{-1/2} \right]^{-3}$$

Then

$$\begin{aligned} L &= \left(\frac{2\,r}{w} \right)^{3/2} Q^{3/4}\, r^{-3/2} \left[r^{-1/2} + 2^{3/2}\, w^{-1/2} \right]^{-3} \\ &= Q^{3/4}\, 2^{3/2}\, w^{-3/2} \left[r^{-1/2} + 2^{3/2}\, w^{-1/2} \right]^{-3} \end{aligned}$$

and finally

$$\begin{aligned} C^*(r, w, Q) &= r\,K + w\,L \\ &= Q^{3/4} \left[r\,r^{-3/2} + w\,2^{3/2}\,w^{-3/2} \right] \left[r^{-1/2} + 2^{3/2}\, w^{-1/2} \right]^{-3} \\ &= Q^{3/4} \left[r^{-1/2} + 2^{3/2}\, w^{-1/2} \right]^{-2} \end{aligned}$$

Question 5: (20 points)

COMMON ERRORS: Major: (1) Wrong expressions for expected utility. Some included p in the utility, etc. (2) Wrong budget constraints, wrong equilibrium conditions, etc. These suggest a basic lack of understanding of the topic. These people should probably stick to six-figure finance. (3) Some people found $p > 1$. That should have been an immediate giveaway that something had gone wrong. Why would anyone before the event pay more than a dollar to get a slip of paper that will pay off only one dollar if the Yankees win and nothing if the Marlins win? Minor: Not saying why the right hand side of the budget constraint is 10000 (simplification of $10000\,p + 10000\,(1-p)$).

WHERE WAS THIS MATERIAL COVERED: You did an almost identical problem in Problem Set 7 Question 3, and a harder one in Problem Set 8, Question 1. The math of Problem Set 5, Question 2 is also identical.

Denote the final wealths in the two scenarios by

$W^{\text{Nina}}_{\text{Yankees}}$ and $W^{\text{Nina}}_{\text{Marlins}}$ for Nina, and $W^{\text{Flora}}_{\text{Yankees}}$ $W^{\text{Flora}}_{\text{Marlins}}$ for Flora.

Then:

(a) (3 points) Expected utility functions

$$\begin{aligned} EU^{\text{Nina}} &= 0.9\ \ln(W^{\text{Nina}}_{\text{Yankees}}) + 0.1\ \ln(W^{\text{Nina}}_{\text{Marlins}}) \\ EU^{\text{Flora}} &= 0.3\ \ln(W^{\text{Flora}}_{\text{Yankees}}) + 0.7\ \ln(W^{\text{Flora}}_{\text{Marlins}}) \end{aligned}$$

(b) (3 points) Budget constraints

$$\begin{aligned} p\ W^{\text{Nina}}_{\text{Yankees}} + (1-p)\ W^{\text{Nina}}_{\text{Marlins}} &\le 10000\ p + 10000\ (1-p) = 10000 \\ p\ W^{\text{Flora}}_{\text{Yankees}} + (1-p)\ W^{\text{Flora}}_{\text{Marlins}} &\le 10000\ p + 10000\ (1-p) = 10000 \end{aligned}$$

(c) (4 points) Demand functions using standard Cobb-Douglas results

$$W^{\text{Nina}}_{\text{Yankees}} = 0.9\,\frac{10000}{p} \qquad W^{\text{Nina}}_{\text{Marlins}} = 0.1\,\frac{10000}{1-p}$$

$$W^{\text{Flora}}_{\text{Yankees}} = 0.3\,\frac{10000}{p} \qquad W^{\text{Flora}}_{\text{Marlins}} = 0.7\,\frac{10000}{1-p}$$

(d) (5 points) Equating the supply and demand for Arrow-Debreu securities for the Yankees-win scenario

$$0.9\,\frac{10000}{p} + 0.3\,\frac{10000}{p} = 10000 + 10000, \qquad \text{or} \qquad 1.2\,\frac{10000}{p} = 20000$$

Therefore $p = 0.6$. By Walras' Law, we don't need to solve the other market clearing condition.
(e) (5 points) Substituting back in the demand function

$$W^{\text{Nina}}_{\text{Yankees}} = 15000 \qquad W^{\text{Nina}}_{\text{Marlins}} = 2500$$

$$W^{\text{Flora}}_{\text{Yankees}} = 7500 \qquad W^{\text{Flora}}_{\text{Marlins}} = 17500$$

In other words, Nina makes a bet whereby she will win \$5000 if the Yankees win and lose \$7500 if the Marlins win; Flora makes the opposite bet. (Poor Nina; having to live in NYC on \$2500!)

Question 6: (30 points)

The grading of the essay is based on a combination of (1) correct and clear statement and explanation of all the concepts, (2) organization of the material, and (3) quality of the writing.

The grading scheme was VERY ROUGHLY as follows:

1) Below 15: at least one of the two concepts is misunderstood or very incomplete.

2) Between 15 and 20: the two concepts are understood and illustrated by examples.

3) Between 20 and 25: more technical results e.g. "without information asymmetries, prices actuarially fair", discussion of separating or pooling equilibria.

4) Between 25 and 30: all of the above plus pertinent graphical illustrations.

On the whole, I find that Arnaud (who graded this question) was somewhat more generous than I would have been.

The following points need to be covered: (1) Definitions of moral hazard and adverse selection, and illustrative examples – shirking of effort, pretending to have higher skill, etc – along the lines of BH Chapter 20, pp. 558–559 and Lecture Handout of Dec. 4, p. 1. (2) Statement that without moral hazard or adverse selection, insurance markets and firms of the kind described would offer actuarially (statistically) fair insurance to everyone, and risk-averse consumers would choose to be fully insured. BH Chapter 19, pp. 532-3 and Lecture Handout of Nov. 25, pp. 3–4. (3) Statement of the effects of moral hazard: less care when insured, no care when fully insured, optimal restriction on insurance contracts. BH Chapter 20, pp. 539-40 and Lecture Handout of Dec. 4, pp. 2–3. (4) Analysis of adverse selection: infeasibility of full insurance for both risk types, separating contracts with restricted insurance for low-risk types, pooling and its infeasibility in equilibrium. BH Chapter 20, pp. 545-7 and Lecture Handout of Dec. 4, pp. 4–8.

Question 7: (30 points)

This is quite similar in its steps etc. to Problem Set 6 Question 2, and the work of Precept Week 8.

COMMON ERRORS: (1) Assuming directly that $P_1 = P_2 = p$ or $Q_1 = Q_2 = q$ under joint profit maximization. (2) Minor – Forgetting to check SOCs for the single-variable maximization problems in (b) and (c). (3) This is not an error as such, but nobody argued directly that the results in (d) and (e) had to be equal, but proved it laboriously.

(a) (2 points) The goods are substitutes: $\partial Q_i / \partial P_j > 0$ for $i \neq j$ (an increase in the price of one increases the quantity demanded of the other, holding its own price constant).

(b) (7 points) With price-setting, the expression for firm 1's profit is

$$\Pi_1 = (P_1 - 5)\ Q_1 = (P_1 - 5)\ (20 - 2\,P_1 + P_2)\,.$$

Therefore

$$\frac{\partial \Pi_1}{\partial P_1} = (20 - 2\,P_1 + P_2) - 2\,(P_1 - 5) = 30 - 4\,P_1 + P_2$$

$$\frac{\partial^2 \Pi_1}{\partial P_1^2} = -4 < 0$$

Therefore firm 1's best response function is given by

$$4\,P_1 = 30 + P_2$$

By symmetry of the functions for the two firms, firm 2's best response function is

$$4\,P_2 = 30 + P_2$$

To solve for the Bertrand-Nash equilibrium, subtract one from the other,

$$4\,(P_1 - P_2) = P_1 - P_2\,, \qquad \text{therefore} \qquad P_1 - P_2 = 0\,.$$

Then each = 10. Substituting in the demand functions, the quantities are 10 each, and therefore profits are $(10 - 5) \times 10 = 50$ each.

(c) (7 points) With quantity-setting, the expression for firm 1's profit is

$$\Pi_1 = (P_1 - 5)\ Q_1 = (15 - \tfrac{2}{3}\,Q_1 - \tfrac{1}{3}\,Q_2)\ Q_1\,.$$

Therefore

$$\frac{\partial \Pi_1}{\partial Q_1} = (15 - \tfrac{2}{3}\,Q_1 - \tfrac{1}{3}\,Q_2) - \tfrac{2}{3}\ Q_1 = 15 - \tfrac{4}{3}\,Q_1 - \tfrac{1}{3}\,Q_2$$

$$\frac{\partial^2 \Pi_1}{\partial Q_1^2} = -\tfrac{4}{3} < 0$$

Therefore firm 1's best response function is given by

$$4\,Q_1 = 45 - Q_2$$

By symmetry of the functions for the two firms, firm 2's best response function is

$$4\,Q_2 = 45 - Q_2$$

To solve for the Cournot-Nash equilibrium, subtract one from the other,

$$4\,(Q_1 - Q_2) = -(Q_1 - Q_2)\,, \qquad \text{therefore} \qquad Q_1 - Q_2 = 0\,.$$

Then each = 9. Substituting in the inverse demand functions, the prices are 11 each, and therefore profits are $(15 - 9) \times 9 = 54$ each.

(d) (6 points) When the firms choose prices jointly to maximize total profit $\Pi = \Pi_1 + \Pi_2$,,

$$\Pi = (P_1 - 5)\ (20 - 2\,P_1 + P_2) + (P_2 - 5)\ (20 + P_1 - 2\,P_2)$$

So

$$\frac{\partial \Pi}{\partial P_1} = (20 - 2\,P_1 + P_2) - 2\,(P_1 - 5) + (P_2 - 5) = 25 - 4\,P_1 + 2\,P_2$$

and similarly

$$\frac{\partial \Pi}{\partial P_1} = 25 + 2\,P_1 - 4\,P_2$$

(Second-order condtions: the matrix of the second-order partial derivatives

$$\begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix}$$

is negative-definite; this is optional.)

Subtracting one FONC from the other establishes $P_1 = P_2$. Then each equals 12.5. Substituting in the demand functions, the quantities are 7.5 each, and therefore the profits for each firm are

$$(12.5 - 5) \times 7.5 = \left(\frac{15}{2}\right)^2 = \frac{225}{4} = 56.25$$

(e) (4 points) When jointly maximizing total profit, the two firms are acting as a cartel or a monopoly, so it makes no difference whether they set prices or quantities. More formally, the demand and inverse demand functions establish a one-to-one relationship between the prices (P_1, P_2) and the quantities (Q_1, Q_2), so anything the firms can get by choosing the former they can get by choosing the latter. (If you didn't recognize this intuition and did the calculation again, no problem.)

(f) (4 points) We see

$$\begin{aligned} &\text{Bertrand Prices} < \text{Cournot Prices} < \text{Collusive Prices} \\ &\text{Bertrand Quantities} > \text{Cournot Quantities} > \text{Collusive Quantities} \\ &\text{Bertrand Profits} < \text{Cournot Profits} < \text{Collusive Profits} \end{aligned}$$

Collusive profit must obviously be greater than that in either non-cooperative Nash equilibrium, because the firms acting together could have chosen the same prices or quantitites as in the Nash equilibrim, and when demands are not independent, they benefit by internalizing their mutual profit-externalities.

When the products are substitutes, the effects (externalities) of one firm's price cut or quantity expansion on the other's profit are negative, therefore in joint-maximization they benefit by cutting prices less or expanding quantities less. This explains the price and quantity comparisons between collusion and either Nash solution.

The difference between Cournot and Bertrand is explained by the fact that each firm faces a less elastic deemand (quantity as a function of its own price) when the other's quantity is held fixex (Cournot) than when the other's price is held fixed (Bertrand). This is the Le Châtelier-Samuelson principle. Therefore each firm has and exercises more monopoly power (raises price or cuts quantity) under Cournot than under Bertrand.