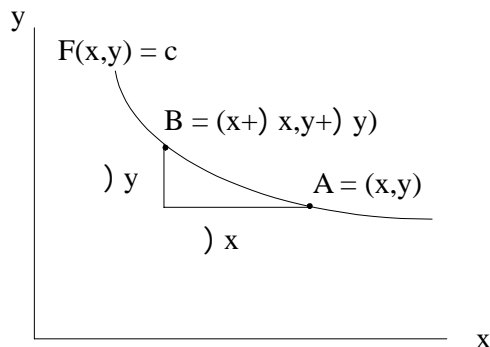


THE IMPLICIT FUNCTION THEOREM

Suppose y is defined as a function of x implicitly by the relationship $F(x, y) = c$, a constant. In (x, y) space, this is a curve; see the figure below. We want to find an expression for the derivative dy/dx , that is, an expression for the slope of the curve at a general point (x, y) on it. Here is a quick-and-dirty but simple way to do it.



Let $A = (x, y)$ be the point, and take a nearby point $B = (x + \Delta x, y + \Delta y)$ on the same curve. The Δx and Δy are small increments. In the figure, the curve slopes downward, so one of them is negative and the other positive. Specifically as shown, $\Delta x < 0$ and $\Delta y > 0$. For different slopes of the curve and different locations of the points A and B along the curve, the increments may have different sign patterns, but the algebra below works out the same way for all cases. If you want to be sure, experiment with some different configurations.

Since both points are on the same curve,

$$F(x + \Delta x, y + \Delta y) = c = F(x, y),$$

or

$$F(x + \Delta x, y + \Delta y) - F(x, y) = 0.$$

By Taylor's theorem for two variables,

$$F(x + \Delta x, y + \Delta y) - F(x, y) = \frac{\partial F}{\partial x} \Delta x + \frac{\partial F}{\partial y} \Delta y + \text{second- and higher-order terms}.$$

Setting this equal to zero, dividing by Δx , and rearranging,

$$\frac{\Delta y}{\Delta x} = - \frac{\partial F / \partial x}{\partial F / \partial y} - \frac{\text{second- and higher-order terms}}{\Delta x \partial F / \partial y}.$$

Letting Δx go to zero, the limit of the left hand side is by definition the derivative along the curve that we want to find, and the second term on the right hand side, having terms of second, third etc. order of smallness in the numerator and only first order in the denominator, goes to 0. Therefore

$$\left. \frac{dy}{dx} \right|_{F(x,y)=c} = - \frac{\partial F / \partial x}{\partial F / \partial y}.$$