INDIRECT UTILITY FUNCTION

$$\begin{aligned} U^*(P_x, P_y, M) &= \max \{ U(x,y) \mid P_x\, x + P_y\, y \le M \} \\ &= U(x^*, y^*) \\ &= U(D_x(P_x, P_y, M), D_y(P_x, P_y, M)) \end{aligned}$$

PROPERTIES OF U^*:

(1) No money illusion – Homogeneous degree zero:

$$U^*(k\,P_x, k\,P_y, kM) = U^*(P_x, P_y, M)$$

(2) As money income changes:

$$\begin{aligned} \frac{\partial U^*}{\partial M} &= \left.\frac{\partial U}{\partial x}\right|_* \frac{\partial x^*}{\partial M} + \left.\frac{\partial U}{\partial y}\right|_* \frac{\partial y^*}{\partial M} \\ &= \lambda \left[P_x \frac{\partial x^*}{\partial M} + P_y \frac{\partial y^*}{\partial M} \right] \\ &= \lambda \frac{\partial M}{\partial M} = \lambda \end{aligned}$$

(3) As price changes:

$$\frac{\partial U^*}{\partial P_x} = \left.\frac{\partial U}{\partial x}\right|_* \frac{\partial x^*}{\partial P_x} + \left.\frac{\partial U}{\partial y}\right|_* \frac{\partial y^*}{\partial P_x}$$

$$
\begin{aligned}
&= \lambda \left[P_x \frac{\partial x^*}{\partial P_x} + P_y \frac{\partial y^*}{\partial P_x} \right] \\
&= -\lambda\, x^* \qquad \text{(just like } M \downarrow \text{ by } x^*\text{)}
\end{aligned}
$$

(Last step: differentiate adding-up identity w.r.t. P_x:

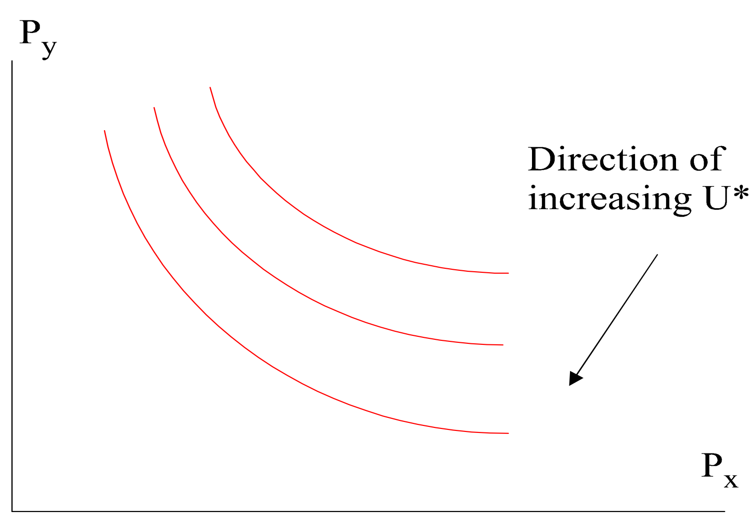
$$P_x\ x^* + P_y\ y^* = M$$

$$x^* + P_x \frac{\partial x^*}{\partial P_x} + P_y \frac{\partial y^*}{\partial P_x} = 0\ )$$

Divide price- and income-change equations :

$$\text{Roy's Identity:}\quad x^* = -\frac{\partial U^* / \partial P_x}{\partial U^* / \partial M}$$

(4) Contours of U^* in (P_x, P_y) space with M fixed:



(Like theater with stage at NE corner)

EXPENDITURE FUNCTION

Solve the indirect utility function for income:

$$u = U^*(P_x, P_y, M) \quad \Longleftrightarrow \quad M = M^*(P_x, P_y, u)$$

$$M^*(P_x, P_y, u) = \min \{ P_x\, x + P_y\, y \,|\, U(x, y) \geq u \}$$

"Dual" or mirror image of utility maximization problem.
Economics – income compensation for price changes
Optimum quantities – Compensated or Hicksian demands

$$x^* = D_x^H(P_x, P_y, u)\,, \qquad y^* = D_y^H(P_x, P_y, u)$$

PROPERTIES OF M^*:

(1) Homogeneous degree 1 in (P_x, P_y) holding u fixed:

$$M^*(k\,P_x, k\,P_y, u) = k\ M^*(P_x, P_y, u)$$

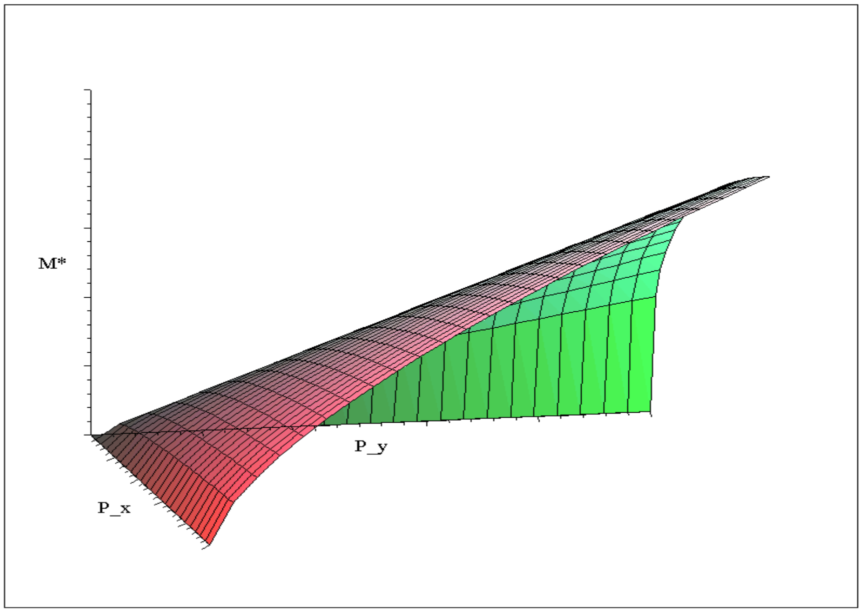
(2) Hotelling's or Shepherd's Lemma –
Compensated demands partial derivatives w.r.t. prices:

$$D_x^H(P_x, P_y, u) = \partial M^* / \partial P_x\,,\ D_y^H(P_x, P_y, u) = \partial M^* / \partial P_y$$

Proof: $M^* = P_x\ D_x^H + P_y\ D_y^H$, $u = U(D_x^H, D_y^H)$. So

$$\begin{aligned}
\partial M^* / \partial P_x &= D_x^H + P_x\ \partial D_x^H / \partial P_x + P_y\ \partial D_y^H / \partial P_x \\
0 &= U_x\ \partial D_x^H / \partial P_x + U_y\ \partial D_y^H / \partial P_x \\
&= \lambda\, [\, P_x\ \partial D_x^H / \partial P_x + P_y\ \partial D_y^H / \partial P_x\,]
\end{aligned}$$

(3) "Weakly" concave in (P_x, P_y) holding u fixed.
Cobb-Douglas example: $(P_x)^{1/3}\ (P_y)^{2/3}$



PROPERTIES OF HICKSIAN DEMAND FUNCTIONS:

(1) Own substitution effect negative:

$$\left.\frac{\partial x}{\partial P_x}\right|_{u=\text{const}} = \frac{\partial D_x^H}{\partial P_x} = \frac{\partial^2 M^*}{\partial P_x^2} \leq 0$$

(2) Symmetry of cross-price effects:

$$\frac{\partial D_x^H}{\partial P_y} = \frac{\partial^2 M^*}{\partial P_x \partial P_y} = \frac{\partial D_y^H}{\partial P_x}$$

(Net) substitutes if > 0, complements if < 0
General concept : Comparative statics

COBB-DOUGLAS EXAMPLE

(Direct) UTILITY FUNCTION:

$$U(x,y) = \alpha\ \ln(x) + \beta\ \ln(y), \qquad \alpha + \beta = 1$$

$$x^* = \alpha\, M/P_x, \qquad y^* = \beta\, M/P_y$$

INDIRECT UTILITY FUNCTION

$$\begin{aligned} U^*(P_x, P_y, M) &= \alpha\ [\ln(\alpha) + \ln(M) - \ln(P_x)\,] \\ &\quad +\beta\ [\ln(\beta) + \ln(M) - \ln(P_y)\,] \\ &= \text{junk} + \ln(M) - \alpha\ \ln(P_x) - \beta\ \ln(P_y) \end{aligned}$$

Roy's Identity:

$$-\ \frac{\partial U^*/\partial P_x}{\partial U^*/\partial M} = -\ \frac{-\,\alpha/P_x}{1/M} = \frac{\alpha\, M}{P_x} = x^*$$

EXPENDITURE FUNCTION

$$M^* = M^*(P_x, P_y, u) = e^u\ (P_x)^\alpha\ (P_y)^\beta$$

Hicksian demand functions

$$x^H = \alpha\ e^u\ (P_x)^{\alpha-1}\ (P_y)^\beta,\ y^H = \beta\ e^u\ (P_x)^\alpha\ (P_y)^{\beta-1}$$

SLUTSKY EQUATION

Link between Marshallian and Hicksian demands

Equal if $u = U^*(P_x, P_y, M)$, $M = M^*(P_x, P_y, u)$.

For good i where i may be either x or y,

$$D_i^H(P_x, P_y, u) = D_i^M(P_x, P_y, M^*(P_x, P_y, u)\,)$$

Now let P_j change, where j may be x or y

$$\begin{aligned}
\frac{\partial D_i^H}{\partial P_j} &= \frac{\partial D_i^M}{\partial P_j} + \frac{\partial D_i^M}{\partial M}\ \frac{\partial M^*}{\partial P_j} \\
&= \frac{\partial D_i^M}{\partial P_j} + \frac{\partial D_i^M}{\partial M}\ D_j^H \\
&= \frac{\partial D_i^M}{\partial P_j} + \frac{\partial D_i^M}{\partial M}\ D_j^M
\end{aligned}$$

For example

$$\left.\frac{\partial x}{\partial P_y}\right|_{u=\text{const}} = \left.\frac{\partial x}{\partial P_y}\right|_{M=\text{const}} + y\ \frac{\partial x}{\partial M}$$

Price derivative of compensated demand $=$

Price derivative of uncompensated demand

$+$ Income effect of compensation.

If $i = j$, LHS is negative. Then Giffen implies Inferior