UNCERTAINTY

Outcomes of actions not perfectly predictable
can differ according to circumstances or "scenarios"
Scenarios or "state of the world": each a complete
description of all economically relevant circumstances
May be a finite list $i = 1$, 2, . . . n, or continuum e.g. [0,1]
Once one scenario transpires, no further uncertainty

An "uncertain prospect" or "lottery" is a
probability distribution of outcomes (wealth, income etc.)
over all possible scenarios:
Probabilities (objective or subjective) π_i, ≥ 0, sum to 1
Can depend on action chosen – safe or gamble etc.
Certainty: only one of the π_i equals 1, all others 0
For any random variable x, with possible values x_i

$$\text{Expected Value: } \mathsf{E}[x] \equiv \overline{x} = \sum_{i=1}^{n} \pi_i \, x_i$$

$$\text{Variance: } \mathsf{V}[x] \equiv \mathsf{E}[\,(x - \overline{x})^2\,] = \sum_{i=1}^{n} \pi_i \, (x_i - \overline{x})^2$$

Risk-neutrality: Concern for $\mathsf{E}[W]$ only.
More generally, trade-off between $\mathsf{E}[W]$ and
$\mathsf{V}[W]$ or some other measure or dispersion

EXPECTED UTILITY

von Neumann - Morgenstern utility function $U(W)$
Preference over uncertain prospects: Expected utility

$$EU[W] = \sum_{i=1}^{n} \pi_i \, U(W_i)$$

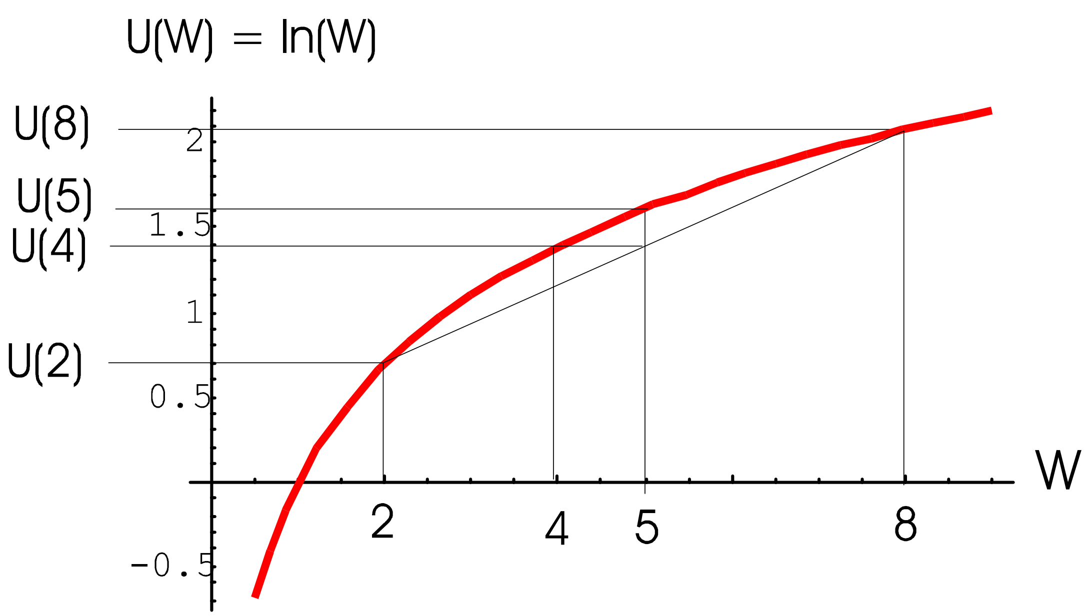
If U is concave, this reflects risk-aversion: example

$$\tfrac{1}{2}\, U(W_1) + \tfrac{1}{2}\, U(W_2) < U(\tfrac{1}{2}\{W_1 + W_2\}\,)$$

e.g. $5 = \frac{1}{2}(2+8)$ but $U(W) = \ln(W)$ gives

$$U(5) > \tfrac{1}{2}\, U(2) + \tfrac{1}{2}\, U(8) = U(4)$$

indifference between sure \$4 and 50:50 chance of \$2 or \$8



Expected utility approach: from set of available actions,
choose the one that yields highest expected utility

When is this a valid representation of preferences?

Usual assumptions (completene, transitive, continuous), PLUS

Independence: Preference between lotteries X, Y same as between $(\pi : X, (1-\pi) : Z)$ and $(\pi : Y, (1-\pi) : Z)$

This can be problematic

If change vN-N utility function $U(W)$ to
increasing linear transform $\widetilde{U}(W) = a + b\,U(W)$

$$\begin{aligned} \mathsf{E}[\widetilde{U}(W)] &= \sum_i \pi_i \widetilde{U}(W_i) = \sum_i \pi_i [\; a + b\; U(W_i) \;] \\ &= a + b \;\sum_i \pi_i U(W_i) = a + b\; \mathsf{E}[U(W)] \end{aligned}$$

So max $\mathsf{E}[U(W)]$ same as max $\mathsf{E}[\widetilde{U}(W)]$.

But if monotone transform *nonlinear:* $\widetilde{U}(W) = \phi(U(W))$

$$\begin{aligned} \mathsf{E}[\widetilde{U}(W)] &= \sum_i \pi_i \widetilde{U}(W_i) = \sum_i \pi_i \phi(U(W_i)) \\ &\neq \phi\left(\sum_i \pi_i U(W_i)\right) = \phi(\mathsf{E}[U(W)]) \end{aligned}$$

Therefore for expected utility approach,
the underlying vN-M utility function must be cardinal

QUANTIFYING RISK-AVERSION

Initial wealth W_0, lottery 50:50 chances of $W_0 \pm x$ (so variance $v = x^2$). Find y such that

$$U(W_0 - y) = \tfrac{1}{2}\, U(W_0 + x) + \tfrac{1}{2}\, U(W_0 - x)$$

Taylor series expansion: $U(W_0) - y\, U'(W_0)$

$$\begin{aligned} &= \tfrac{1}{2}\, [\, U(W_0) + x\, U'(W_0) + \tfrac{1}{2}\, x^2\, U''(W_0)] \\ &\quad + \tfrac{1}{2}\, [\, U(W_0) - x\, U'(W_0) + \tfrac{1}{2}\, x^2\, U''(W_0)] \\ &= U(W_0) + \tfrac{1}{2}\, x^2\, U''(W_0) \end{aligned}$$

Therefore mean-variance tradeoff for small risk around W_0:

$$\frac{y}{x^2} = \frac{1}{2}\; \frac{-U''(W_0)}{U'(W_0)} = \tfrac{1}{2}\; A(W_0)$$

where $A(W_0)$ is called the *absolute risk aversion*.

For proportional risks $W_0\,(1 \pm x)$:

$$U(W_0\,(1 - y)\,) = \tfrac{1}{2}\; U(W_0(1 + x)\,) + \tfrac{1}{2}\; U(W_0(1 - x)\,)$$

$$\frac{y}{x^2} = \frac{1}{2}\; \frac{-W_0\; U''(W_0)}{U'(W_0)} = \tfrac{1}{2}\; R(W_0)$$

where $R(W_0)$ is called the *relative risk aversion*.

Constant absolute risk aversion (used in finance):

$$U(W) = -\exp(-A\,W)$$

Constant relative risk aversion:

$$U(W) = \begin{cases} W^{1-R}/(1-R) & \text{if } R \neq 1 \\ \ln(W) & \text{if } R = 1 \end{cases}$$

For this case with $R \neq 1$ expected utility has CES form:

$$EU[W] = \frac{1}{1-R} \sum_i \pi_i \, (W_i)^{1-R}$$

If $R = 1$ it has Cobb-Douglas form

$$EU[W] = \sum_i \pi_i \, \ln(W_i)$$

Example: Initial wealth
$W_0 =$ \$1 million,
Risk: 50:50 of \$100,000
so proportional $x = 0.1$
Table of \$ $W_0\,y$
you would pay to avoid risk :

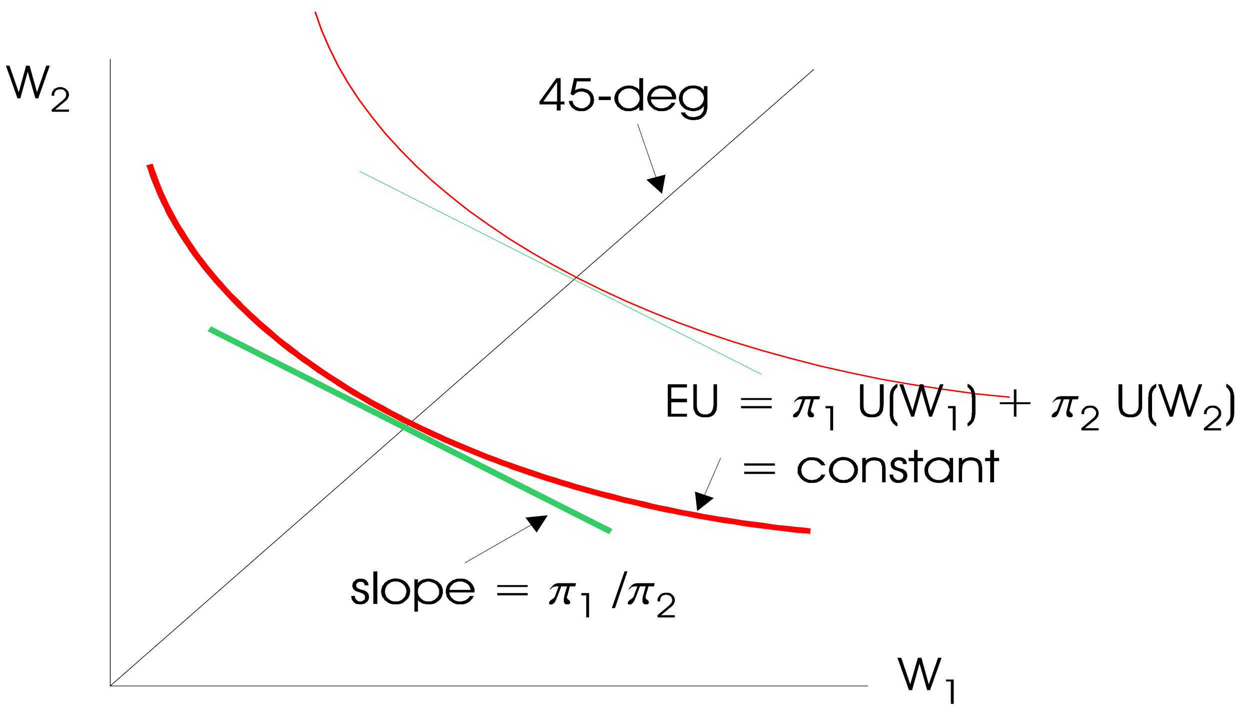
R	$W_0\,y$
0.5	2,500
1	5,000
2	10,000
3	14,900
10	44,200
20	67,600

STATE-PREFERENCE APPROACH

Two scenarios (for geometric illustration), $\pi_1 + \pi_2 = 1$

$$EU[W] = \pi_1\, U(W_1) + \pi_2\, U(W_2)$$

Contours of expected utility:



$$MRS_{21} = - \left.\frac{dW_2}{dW_1}\right|_{EU=\text{const}} = \frac{\partial EU / \partial W_1}{\partial EU / \partial W_2} = \frac{\pi_1}{\pi_2}\,\frac{U'(W_1)}{U'(W_2)}$$

Diminishing MRS $\equiv$ Risk-aversion ($U'' < 0$)

Along 45-degree line, $W_1 = W_2$, so no risk
and $MRS_{21} = \pi_1 \,/\, \pi_2$