PORTFOLIO CHOICE

One Riskless, One Risky Asset

Safe asset: gross return rate R (1 plus interest rate)

Risky asset: random gross return rate r

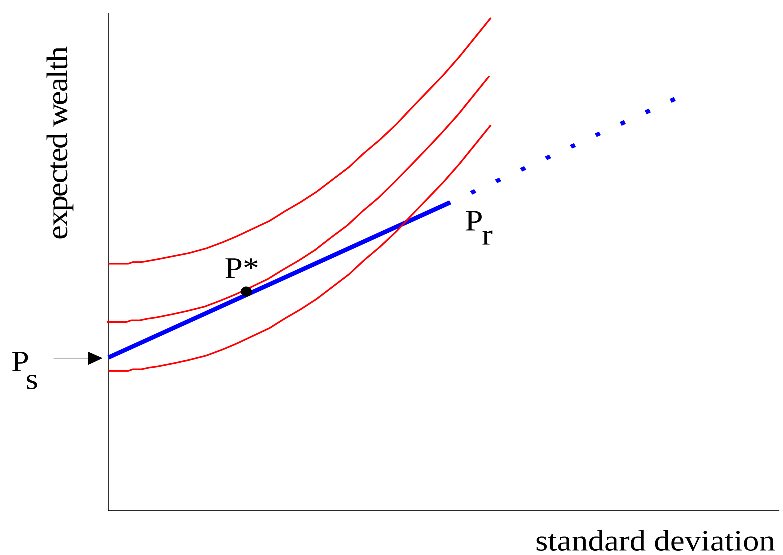
Mean $\mu = \mathsf{E}[r] > R$, Variance $\sigma^2 = \mathsf{V}[r]$

Initial wealth W_0. If x in risky asset,

final wealth $W = (W_0 - x)\,R + x\,r = R\,W_0 + (r - R)\,x$

$$\begin{aligned} \mathsf{E}[W] &= W_0\,R + x\,(\mu - R) \\ \mathsf{V}[W] &= x^2\,\sigma^2; \quad \text{Std. Dev.} = x\,\sigma \end{aligned}$$

As x varies, straight line in (Mean,Std.Dev.) figure.



$P_s = (0, W_0\,R)$ safe; $P_r = (W_0\,\sigma, W_0\,\mu)$ risky;

Beyond P_r possible if leveraged borrowing OK

Objective function Mean $-\ a$ (Std.Dev.)^2; so P^* optimal

Two Risky Assets

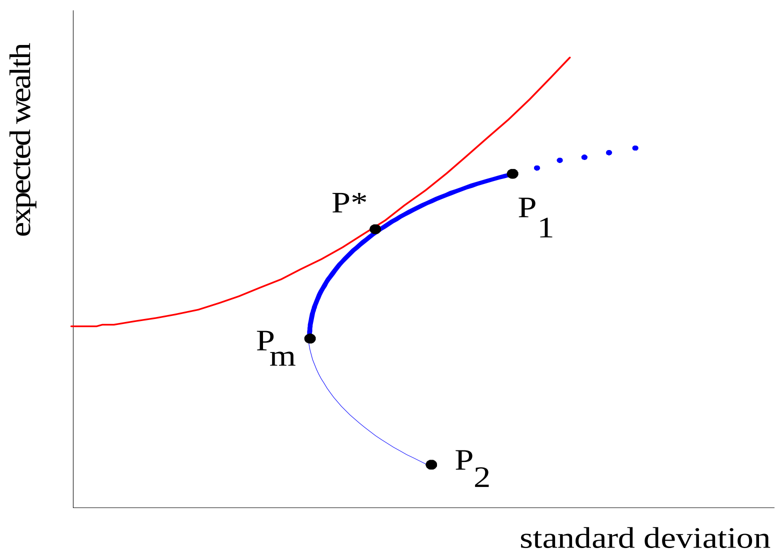
$W_0 = 1$; Random gross return rates r_1, r_2

Means $\mu_1 > \mu_2$; Std. Devs. σ_1, σ_2, Correl. Coefft. ρ

Portfolio $(x, 1-x)$. Final $W = x\, r_1 + (1-x)\, r_2$

$$\mathsf{E}[W] = x\,\mu_1 + (1-x)\,\mu_2 = \mu_2 + x\,(\mu_1 - \mu_2)$$

$$\mathsf{V}[W] = x^2\,(\sigma_1)^2 + (1-x)^2\,(\sigma_2)^2 + 2\,x\,(1-x)\,\rho\,\sigma_1\,\sigma_2$$
$$= (\sigma_2)^2 - 2\,x\,\sigma_2\,(\sigma_2 - \rho\,\sigma_1) + x^2\,[(\sigma_1)^2 - 2\,\rho\,\sigma_1\,\sigma_2 + (\sigma_2)^2]$$



Diversification can reduce variance if $\rho < \min\,[\sigma_1/\sigma_2, \sigma_2/\sigma_1]$

P_1, P_2 points for the two individual assets

P_m minimum-variance portfolio

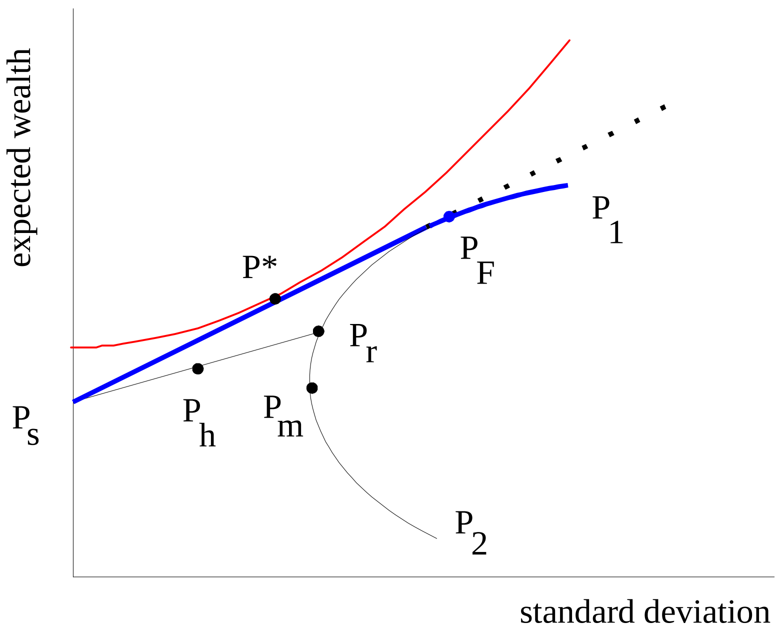
Portion $P_2\,P_m$ dominated; $P_m\,P_1$ efficient frontier

Continuation past P_1 if short sales of 2 OK

Optimum P^* when preferences as shown

One Riskless, Two Risky Assets

First combine two riskies; then mix with riskless



This gets all points like P_h on all lines like $P_s\,P_r$
Efficient frontier $P_s\,P_F$ tangential to risky combination curve
Then along curve segment $P_F\,P_1$ if no leveraged borrowing;
continue straight line $P_s\,P_F$ if leveraged borrowing OK

With preferences as shown, optimum P^*
mixes safe asset with particular risky combination P_F
"Mutual fund" P_F is the same for all investors
regardless of risk-aversion (so long as optimum in $P_s\,P_F$)

Even less risk-averse people may go beyond P_F
including corner solution at P_1
or tangency past P_1 if can sell 2 short to buy more 1

CAPITAL ASSET PRICING MODEL

Individual investors take the rates of return as given
but these must be determined in equilibrium
Add supply side – firms issue equities
Take production, profit-max as exogenous
Two firms, profits Π_1 and Π_2. Means $\mathsf{E}[\Pi_1]$, $\mathsf{E}[\Pi_2]$;
Variances $\mathsf{V}[\Pi_1]$, $\mathsf{V}[\Pi_2]$; Covariance $\mathsf{Cov}[\Pi_1, \Pi_2]$
Safe asset (government bond) sure gross return rate R

Market values of firms F_1, F_2; to be solved for (endogenous)
(Random) rates of return $r_1 = \Pi_1 / F_1$ and $r_2 = \Pi_2 / F_2$,
and for whole market, $r_m = (\Pi_1 + \Pi_2)/(F_1 + F_2)$
After a lot of algebra, important results:

$$(1) \quad \mathsf{E}[r_1] - R = \frac{\mathsf{Cov}[r_1, r_m]}{\mathsf{V}[r_m]} \; \{ \; \mathsf{E}[r_m] - R \; \}$$

Risk premium on firm-1 stock depends on its
systematic risk (correlation with whole market) only,
not *idiosyncratic* risk (part uncorrelated with market)
Coefficient is *beta* of firm-1 stock

$$(2) \quad F_1 = \frac{\mathsf{E}[\Pi_1] - A \; \mathsf{Cov}[\Pi_1, \Pi_1 + \Pi_2]}{R}$$

where A is the market's aggregate risk-aversion (usually small)

Value of firm = present value of its profits
adjusted for systematic risk, and
discounted at riskless rate of interest

ROCKET-SCIENCE FINANCE

Equity, debt etc - complex pattern of payoffs in
different scenarios: vector $S = (S_1, S_2, \ldots)$

Owning security S is full equivalent to
owning portfolio of Arrow-Debreu securities (ADS):
S_1 of ADS_1, S_2 of ADS_2, ...

In equilibrium, no "riskless arbitrage" profit available
So relation bet. price P_S of S and ADS prices p_i:

$$P_S = S_1\, p_1 + S_2\, p_2 + \ldots$$

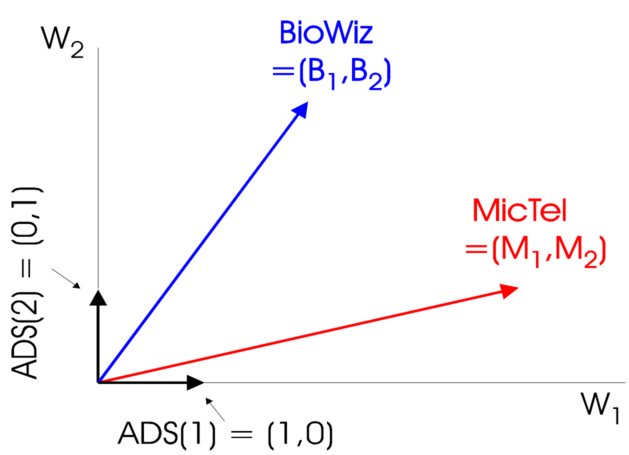
Converse example: Two scenarios, two firms's shares payoff
MicTel (M_1, M_2), BioWiz (B_1, B_2).

If X_M of MicTel + X_B of BioWiz $\equiv$ 1 of ADS_1,

$$X_M\ M_1 + X_B\ B_1 = 1, \quad X_M\ M_2 + X_B\ B_2 = 0$$

$$X_M = \frac{B_2}{M_1\, B_2 - B_1\, M_2}, \quad X_B = \frac{-M_2}{M_1\, B_2 - B_1\, M_2}$$

One of these may be negative: need short sales



ADS's can be "constructed" from available securities
Then no-arbitrage-in-equilibrium condition:

$$P_1 = X_M \; P_{\text{MicTel}} + X_B \; P_{\text{BioWiz}}$$

Similarly P_2. So the "constructed" ADS's can be priced.

Every financial asset is defined by its vector of payoffs in all scenarios. Therefore it can be priced using these prices of all ADS's ("pricing kernel")
Examples – options and other derivatives

General idea: Markets for risks are complete, and achieve Pareto-efficient allocation of risks if enough securities exist that their payoff vectors span the space of wealths in all scenarios

"Rocket-science finance" extends this idea to infinite-dimensional spaces of scenarios
If sequence of periods, need enough markets to span the scenarios one-period ahead, and then rebalance portfolio by trade (dynamic hedging)

Finance = General equilibrium + Linear algebra !

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