ECO 305 – Fall 2003 – Avinash Dixit

Financial Markets

1 What Financial Markets Do

Financial markets perform two important functions. One is to enable transactions across time by bringing together borrowers and lenders. We can regard this in the standard framework of the general equilibrium of exchange and production. We have already seen how the intertemporal choices of consumers and firms can be modeled using discounted present values. These decisions are then functions of the interest rates; it only remains to find the equilibrium structure of interest rates across different periods to balance all these plans. This is just a particular interpretation of general competitive equilibrium – we label goods available at different times as economically distinct commodities, and the relative price P_t/P_{t+1} is just 1 plus the interest rate between these two periods. Intermediate macro courses study such *intertemporal choice and equilibrium* in detail, and build theories of saving, investment, and growth.

The second function of financial markets is to enable people to trade risk. Left to themselves, some people may face more risk or a different kind of risk, and different people may have different degrees of tolerance for risk. Financial markets enable them to trade risks with others, until an equilibrium allocation of risks is achieved that is preferred by all to the no-trade situation. Such markets for risk can also be fitted into the framework of general equilibrium analysis. But this is not treated in detail in intermediate micro books, so I will spend more time on it.

Our introduction to the theory of financial markets proceeds along two tracks. First comes some simple theory of portfolio selection and its implications for the pricing of capital assets and of risk, using an approach where people value just two aspects of their portfolio – the probabilistic average or "mathematical expectation" or the probabilistic average return (which they like), and the variance or standard deviation of this return (which they dislike). The second approach is at a deeper conceptual level, and obtains the general equilibrium pricing of wealth in different states of the world (scenarios) as governed by the risks and risk-preferences of all participants in the market. Roughly speaking, the first will help you appear an intelligent salesperson when talking to a client; the second will equip you to become a financial analyst who can invent and price new innovative derivative securities. Of course in each case we make only a beginning toward these aims; you will have to acquire many further details from specialized finance courses.

2 Portfolio Choice

In this section we consider the asset choices of an investor who likes a high average rate of return but dislikes risk. In the next section we use this to study the equilibrium of a market that brings together many such investors. We begin with the simplest case and gradually work our way to more general ones.

2.1 One Riskless and One Risky Asset

Suppose the investor has initial wealth W_0, and can invest it in either a safe asset that pays a gross rate of return (principal plus interest) of R per dollar invested, or a risky asset which pays a random gross rate of return r per dollar invested. Let $\mu = \mathrm{E}[r]$ denote the expected value of this random rate of return, and $\sigma^2 = \mathrm{V}[r]$ its variance. We assume $\mu > R$; if the safe asset had a higher sure rate of return than the expected return on the risky asset, then a risk-averse investor would hold none of the latter.

Suppose the investor puts x dollars in the risky asset, and $W_0 - x$ in the safe asset. Then his final wealth will be

$$W = (W_0 - x)\, R + x\, r\,.$$

This is a random variable, with the expected value

$$\mathrm{E}[W] = (W_0 - x)\, R + x\, \mu = W_0\, R + x\, (\mu - R)\,. \tag{1}$$

We can think of $(\mu - R)$ as the expected rate of excess return from the risky asset (in excess of, or over and above, that on the riskless asset). The variance of wealth is

$$\mathrm{V}[W] = x^2\,\sigma^2 \tag{2}$$

The standard deviation of wealth is $S = x\,\sigma$.

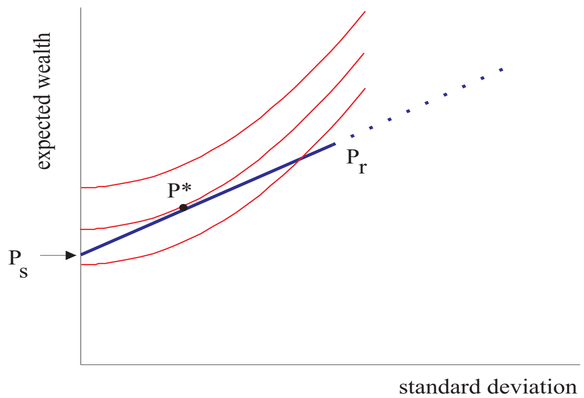

Figure 1: One Safe and One Risky Asset

Figure 1 shows the expected value on the vertical axis and the standard deviation on the horizontal axis. The point P_s has coordinates $(0, W_0\,R)$; here $x = 0$ and the investor plays it entirely safe. The point P_r has coordinates $(W_0\,\sigma, W_0\,\mu)$; here $x = W_0$ so the investor puts all his money in the risky asset. The line has positive slope because $\mu > R$. The dotted continuation of the line beyond P_r can be interpreted as $x > W_0$. Then $W_0 - x < 0$, so the investor holds negative amounts of the safe asset, that is, borrows at the safe rate R to put more than his own wealth into the risky asset, if anyone will lend him money for such "leveraged" risky investment.

The figure also shows the investor's indifference map between the expected return and the risk as measured by the standard deviation of the return. The investor's utility increases toward the north-west (as the expected return increases and the risk decreases); therefore the indifference curves are upward-sloping. The convexity is the usual diminishing marginal rate of substitution. The point of tangency P^* shows the investor's optimal portfolio composition. If the investor were insufficiently risk-averse, the tangency could be beyond P_r. Then the optimum would involve borrowing at a safe rate to make the extra risky investment if that is possible. If not, there will be a solution at the end-point P_r.

Algebraic analysis adds precision to this. Suppose the expected utility function is

$$EU = \mathrm{E}[W] - \tfrac{1}{2}\,\alpha\,\mathrm{V}[W]\,, \tag{3}$$

where α measures the investor's dislike of risk (variance of wealth) in relation to his liking for return (expected wealth). In this case, an indifference curve, namely the locus of all those combinations of expected wealth and standard deviations for which expected utility is some constant k, is given by

$$\text{Expected wealth} = k + \tfrac{1}{2}\,\alpha\,(\text{Standard deviation})^2\,,$$

which is a parabola. Its curvature is higher when the measure or risk aversion α is higher. And the indifference map (the whole set of curves for different constants k) is found by shifting the parabola vertically upward or downward.

Using the expressions for the return and the risk for the portfolio in this example, expected utility becomes

$$EU = W_0 \, R + x \, (\mu - R) - \tfrac{1}{2} \, \alpha \, \sigma^2 \, x^2 \, .$$

Then

$$\frac{d \, EU}{dx} = (\mu - R) - \alpha \, \sigma^2 \, x$$

So long as $\mu > R$, the derivative is positive at $x = 0$, so there can't be a boundary solution at $x = 0$; the investor is willing to take at least a little bit of risk so long as the expected rate of excess return is positive. If leveraged investment is possible, then the solution is simply

$$x = (\mu - R)/(\alpha \, \sigma^2) \, .$$

If leveraged investment is not possible, then this solution is valid only if it is less than W_0; otherwise the derivative $d \, EU/dx$ is positive at $x = W_0$ and we have a boundary solution where the investor puts everything into the risky asset.

Before we go on to more general portfolio choice problems, let us look a little more closely at the mean-variance utility function (3), which we simply postulated. Although it looks reasonable, can it be derived from the basic framework of choice under uncertainty we have used, namely expected utility? That turns out to require some special structure. Suppose the investor's von Neumann–Morgenstern utility function has the "constant absolute risk aversion" form

$$U(W) = - \exp(-\alpha \, W) \, ,$$

where α is the measure of (absolute) risk-aversion. Suppose also that the random wealth W has a normal distribution with mean $\mathrm{E}[W]$ and variance $\mathrm{V}[W]$. Then some algebraic gymnastics can be performed using the normal density function to show that the expected utility is

$$\begin{aligned} \mathrm{E}[U(W)] &= - \exp\big\{ -\alpha \, \mathrm{E}[W] + \tfrac{1}{2} \, \alpha^2 \, \mathrm{V}[W] \, \big\} \\ &= - \exp\big\{ -\alpha \, (\mathrm{E}[w] - \tfrac{1}{2} \, \alpha \, \mathrm{V}[W]) \, \big\} \end{aligned}$$

Maximizing this is equivalent to maximizing the function (3) above.

2.2 Two Risky Assets

The simple model can be readily generalized. Begin with two risky assets. For simplicity of notation, choose the unit of money so that $W_0 = 1$. Let asset i have a random gross rate of return r_i with expected value μ_i and variance σ_i^2, for $i = 1$ and 2. Label the assets so that $\mu_1 > \mu_2$. We must also allow the possibility that these two random return are correlated; let ρ be the correlation coefficient.

Suppose the investor puts x dollars in asset 1, and $1 - x$ dollars in asset 2. His random final wealth is

$$W = x \, r_1 + (1 - x) \, r_2 \, ,$$

with mean

$$\mathrm{E}[W] = x \, \mu_1 + (1 - x) \, \mu_2 = \mu_2 + x \, (\mu_1 - \mu_2) \tag{4}$$

and variance

$$\begin{aligned} \mathrm{V}[W] &= x^2 \, (\sigma_1)^2 + (1 - x)^2 \, (\sigma_2)^2 + 2 \, x \, (1 - x) \, \rho \, \sigma_1 \, \sigma_2 \\ &= (\sigma_2)^2 - 2 \, x \, \sigma_2 \, (\sigma_2 - \rho \, \sigma_1) + x^2 \, [(\sigma_1)^2 - 2 \, \rho \, \sigma_1 \, \sigma_2 + (\sigma_2)^2] \end{aligned} \tag{5}$$

where ρ is the *coefficient of correlation* between the rates of return on the two assets; it will play a crucial role in all that follows.

Figure 2 shows the mean and standard deviation for all possible portfolios. The point $P_1 = (\sigma_1, \mu_1)$ is where the investor puts all his wealth into asset 1, and $P_2 = (\sigma_2, \mu_2)$ where he puts all his wealth into asset 2.

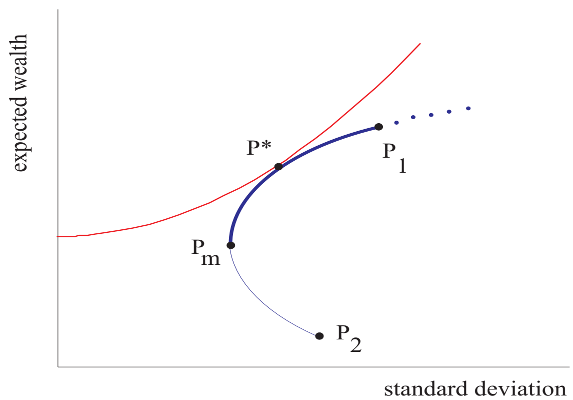

Figure 2: Two Risky Assets

We have labeled the assets so that $\mu_1 > \mu_2$, and then it is natural to think that $\sigma_1 > \sigma_2$ so there would be a risk-return tradeoff. The figure shows this case. But most interestingly, a tradeoff can arise in various combinations of asset-holdings in the portfolio, even if $\sigma_1 < \sigma_2$.

See how the variance behaves as a function of x. Differentiating (5), we have

$$\frac{d\mathrm{V}[W]}{dx} = -2\,\sigma_2\,(\sigma_2 - \rho\,\sigma_1) + 2\;x\,[(\sigma_1)^2 - 2\,\rho\,\sigma_1\,\sigma_2 + (\sigma_2)^2]$$

This is negative at $x = 0$ if $\sigma_2 > \rho\,\sigma_1$, and it is positive at $x = 1$ if $\sigma_1 > \rho\,\sigma_2$. Both these things are true if

$$\rho < \min\,[\sigma_1/\sigma_2, \sigma_2/\sigma_1]$$

Therefore so long as the random returns on the assets are not too positively correlated, that is, there is enough probability that one asset pays off well when the other pays off badly and vice versa, the variance of the total portfolio is minimized for some value of x between 0 and 1, that is, by diversification. If the two asset returns are uncorrelated, that is $\rho = 0$, then the condition for diversification is obviously met; but diversification remains desirable even if there is a little positive correlation.

In the figure, the variance is minimum when the portfolio proportions correspond to the point P_m. The whole range between this portfolio and the point P_2 representing the single asset with the lower expected return is worse than P_m in both the dimensions of expected return and risk; no investor will choose a proportion of asset 2 less than that at P_m. There is a risk-return tradeoff in the range between the minimum-variance portfolio point P_m and the asset with the higher expected return (the point P_1). Investors with appropriate risk-return tradeoffs will find their optimum portfolios in this range; the figure shows the indifference map and the optimum P^* for one such investor. (If risk-aversion is too low, and the person can put more than 100 percent of wealth in one asset by selling the other short, the solution may be beyond P_1. This is shown by the dotted continuation of the curve beyond P_1.)

2.3 One Safe and Two Risky Assets

Next suppose there is one safe asset and two risky assets. We can analyze this case by combining the methods of the two cases considered before. Think of any portfolio combining all three assets as made up of two, the riskless asset and a risky asset which is itself a composite of the two risky assets originally given. Figure 3 shows the process. The mean and standard deviation of the composite risky asset is a point, say P_r, along the curve $P_1 P_m P_2$; the exact location of P_r depends on the proportions with which it combines assets 1 and

2. The mean and standard deviation of the portfolio that includes the safe asset P_s and the composite risky asset P_r is then along the straight line joining P_s and P_r. For example, suppose the point P_r shown in the figure represents the risky portfolio with 60% of wealth in P_1 and 40% in P_2; then the midpoint P_h of the line P_sP_r represents the portfolio that has 50% in the safe asset, 30% in P_1 and 20% in P_2. Varying the proportions of the two risky assets in the mixture represented by P_r, and the proportions between the safe asset and P_r, traces out all possible portfolios. The efficiency frontier of all such portfolio is shown as the thick curve $P_sP_FP_1$. The segment from P_s to P_F is a straight line, and it is tangential at the point P_F to the curve showing the risky portfolios. Thereafter it proceeds along the risky portfolio curve to P_1 if leveraged borrowing to invest more than 100% of one's wealth in the risky asset is not possible. If such borrowing is possible, then the frontier remains a straight line that is a continuation of P_sP_F; this is shown dotted. The optimum can be found by tangency with the indifference map as before. The figure shows this occuring at the point P^* in the segment P_sP_F, where the investor allocates his initial wealth partly to the safe asset and partly to the risky combination represented by P_F.

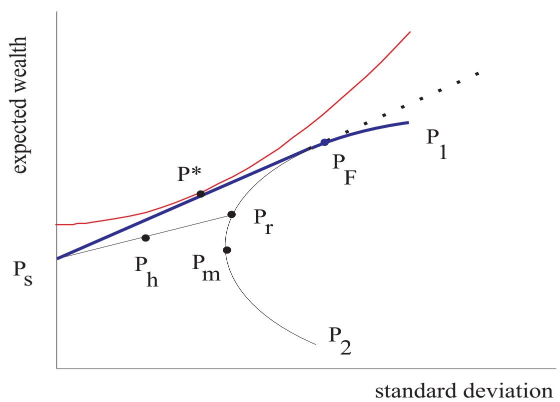

Figure 3: One Safe and Two Risky Assets

The portfolio P_F plays a special role. The investors whose optimum occurs along the straight line segment P_sP_F (and in fact all investors if safe borrowing to invest risky is permitted) choose the same composite risky portfolio, namely P_F. They combine this with the safe asset in varying proportions depending on their risk-aversion. But the *relative* proportions of the two risky assets are the same irrespective of risk-aversion. Therefore instead of marketing the two risky assets separately, suppliers of these assets can provide (or financial intermediaries can package) just one risky asset, namely the "mutual fund" represented by P_F, and leave investors to mix between the safe asset and this mutual fund. This idea, that the large collection of possible risky assets can be reduced to a small number of mutual funds, is important in financial economics more generally.

3 The Capital Asset Pricing Model (CAPM)

In the previous section we studied an individual investor's portfolio choice problem, assuming that he takes as given (exogenous) the rates of return on the available assets. This is the "demand side" of the market. We must now build up a supply side, and then put the two together into a market equilibrium that will determine these rates of return. On the supply side, we posit that the securities are equities issued by firms. We conceal the details of their production functions and profit maximization, which we had studied earlier, and look only at the resulting profits. The previous theory was deterministic. Now we recognize that the profits are also affected by other random influences, and are therefore uncertain. To keep things simple, suppose there are two firms, whose profits are denoted by Π_1 and Π_2. These are random variables; we denote their expected values by $\mathrm{E}[\Pi_1]$ and $\mathrm{E}[\Pi_2]$, and their variances by $\mathrm{V}[\Pi_1]$ and $\mathrm{V}[\Pi_2]$. The two firms' fates may also be correlated, positively or negatively, and we denote the covariance between their profits by $\mathrm{Cov}[\Pi_1, \Pi_2]$.

There is also a safe asset, such as a government bond, that pays a sure gross return at rate R. Investors can allocate their portfolios between this asset and shares in the two firms.

The ultimate aim is to determine the valuations of the two firms in a market equilibrium. Denote these by F_1 and F_2. To make better contact with the demand side (investors' portfolio choice), note that the rates of return per dollar invested in these firms will be $r_1 = \Pi_1/F_1$ and $r_2 = \Pi_2/F_2$ respectively. Individual investors will take r_1 and r_2 as given. These are random variables, and

$$\mathrm{E}[r_i] = \frac{\mathrm{E}[\Pi_i]}{F_i}, \quad \mathrm{V}[r_i] = \frac{\mathrm{V}[\Pi_i]}{(F_i)^2}, \quad \mathrm{Cov}[r_1, r_2] = \frac{\mathrm{Cov}[\Pi_1, \Pi_2]}{F_1\, F_2} \tag{6}$$

We will also find it useful to define the random rate of return on the total of the risky assets. Since in equilibrium all these assets (ownership rights in both firms) are held by someone, we can call this the *market rate of return*

$$r_m = \frac{\Pi_1 + \Pi_2}{F_1 + F_2} \tag{7}$$

and then

$$\mathrm{E}[r_m] = \frac{\mathrm{E}[\Pi_1] + \mathrm{E}[\Pi_2]}{F_1 + F_2}, \quad \mathrm{V}[r_m] = \frac{\mathrm{V}[\Pi_1 + \Pi_2]}{(F_1 + F_2)^2} = \frac{\mathrm{V}[\Pi_1] + 2\ \mathrm{Cov}[\Pi_1, \Pi_2] + \mathrm{V}[\Pi_2]}{(F_1 + F_2)^2} \tag{8}$$

Finally, we will need the covariance between the rates of return to one of the firms' shares and the market as a whole:

$$\mathrm{Cov}[r_i, r_m] = \frac{\mathrm{Cov}[\Pi_1, \Pi_1 + \Pi_2]}{F_1\,(F_1 + F_2)} = \frac{\mathrm{V}[\Pi_1] + \mathrm{Cov}[\Pi_1, \Pi_2]}{F_1\,(F_1 + F_2)} \tag{9}$$

The algebra of the next page or so is rather complicated. I do not expect you to remember or reproduce it on an exam. But I do expect you to read and understand the steps of the argument, and above all, to remember the final results – equations (14), (15), and (16) – and to know their economic interpretations.

Consider an investor h with initial wealth W_h and a mean-variance utility function with coefficient of absolute risk aversion α_h, that is,

$$\mathrm{E}[W] - \tfrac{1}{2}\,\alpha_h\,\mathrm{V}[W]$$

where W is the random final wealth. If he invests x_1^h dollars in the shares of firm 1, x_2^h dollars in the shares of firm 2, and $(W_h - x_1^h - x_2^h)$ in the safe asset, then

$$W = (W_h - x_1^h - x_2^h)\,R + x_1^h\,r_1 + x_2^h\,r_2 = W_h\,R + x_1^h\,(r_1 - R) + x_2^h\,(r_2 - R)\,,$$

and

$$\mathrm{E}[W] = W_h\,R + x_1^h\,(\mathrm{E}[r_1] - R) + x_2^h\,(\mathrm{E}[r_2] - R), \quad \mathrm{V}[W] = (x_1^h)^2\ \mathrm{V}[r_1] + 2\,x_1^h\,x_2^h\,\mathrm{Cov}[r_1, r_2] + (x_2^h)^2\ \mathrm{V}[r_2]\,.$$

Substituting this in the expression for utility and differentiating, the FONCs for the optimal portfolio choice are

$$\begin{aligned} \mathrm{E}[r_1] - R &= \alpha_h\ \{\ x_1^h\,\mathrm{V}[r_1] + x_2^h\,\mathrm{Cov}[r_1, r_2]\ \} \\ \mathrm{E}[r_2] - R &= \alpha_h\ \{\ x_1^h\,\mathrm{Cov}[r_1, r_2] + x_2^h\,\mathrm{V}[r_1]\ \} \end{aligned} \tag{10}$$

These can be solved for x_1^h and x_2^h, the investor's holdings of the risky assets. The solution will depend on his coefficient of absolute risk aversion α_h, but not on his initial wealth W_h. This is like the no-income-effects property of quasi-linear utility functions in choice under uncertainty. Just as quasi-linear utility functions were a convenient tool for analyzing one sector of the economy in isolation (partial equilibrium analysis), mean-variance or constant absolute risk aversion utility functions are convenient (but not always realistic) for analysis of the risky sector on its own.

Rewrite these equations as

$$\begin{aligned} \tau_h \ \{ \, \mathrm{E}[r_1] - R \, \} &= x_1^h \, \mathrm{V}[r_1] + x_2^h \, \mathrm{Cov}[r_1, r_2] \\ \tau_h \ \{ \, \mathrm{E}[r_2] - R \, \} &= x_1^h \, \mathrm{Cov}[r_1, r_2] + x_2^h \, \mathrm{V}[r_1] \end{aligned} \tag{11}$$

where τ_h is the reciprocal of α_h, and can be called the investor's *risk-tolerance*. Add these equations across all investors, and note that in equilibrium the sum of their holdings of each firm's shares must equal the total values of the shares. Therefore

$$\begin{aligned} T\{ \, \mathrm{E}[r_1] - R \, \} &= F_1 \, \mathrm{V}[r_1] + F_2 \, \mathrm{Cov}[r_1, r_2] \\ T\{ \, \mathrm{E}[r_2] - R \, \} &= F_1 \, \mathrm{Cov}[r_1, r_2] + F_2 \, \mathrm{V}[r_1] \end{aligned} \tag{12}$$

where T is the sum of all the τ_hs, and can be called the *market's risk tolerance*. With numerous investors, even though each may be very risk averse, the sum will make the market as a whole much more risk-tolerant.

Multiply the first of these equations by F_1, the second by F_2 and add them:

$$T \ \{ \ (F_1 \, \mathrm{E}[r_1] + F_2 \, \mathrm{E}[r_2]) - R \, (F_1 + F_2) \ \} = (F_1)^2 \, \mathrm{V}[r_1] + 2 \, F_1 \, F_2 \, \mathrm{Cov}[r_1, r_2] + (F_2)^2 \, \mathrm{V}[r_2] \, .$$

Using (6), this becomes

$$T \ \{ \, \mathrm{E}[\Pi_1 + \Pi_2] - R \, (F_1 + F_2) \ \} = \mathrm{V}[\Pi_1] + 2 \, \mathrm{Cov}[\Pi_1, \Pi_2] + \mathrm{V}[\Pi_2] = \mathrm{V}[\Pi_1 + \Pi_2]$$

Next, using (8),

$$T \ (F_1 + F_2) \ \{ \ \mathrm{E}[r_m] - R \ \} = (F_1 + F_2)^2 \ \mathrm{V}[r_m]$$

or

$$\mathrm{E}[r_m] - R = \frac{F_1 + F_2}{T} \ \mathrm{V}[r_m] \tag{13}$$

This gives the excess of the expected rate of return on the risky assets as a whole over the safe rate of return, or the risk premium on the market as a whole. It is proportional to the variance of the market rate of return, and inversely proportional to the market's risk tolerance. The factor on the right hand side multiplying the variance, namely $(F_1 + F_2)/T$, is called the *market price of risk*. It depends on the values of the firms and is therefore a property of the whole equilibrium.

Now take the first of the market equilibrium equations (12) and multiply it by F_1:

$$\begin{aligned} T \, F_1 \, \{ \, \mathrm{E}[r_1] - R \, \} &= (F_1)^2 \, \mathrm{V}[r_1] + F_1 \, F_2 \, \mathrm{Cov}[r_1, r_2] \\ &= \mathrm{V}[\Pi_1] + \mathrm{Cov}[\Pi_1, \Pi_2] \\ &= F_1 \, (F_1 + F_2) \ \mathrm{Cov}[r_1, r_m] \end{aligned}$$

where to get the last line I have used (9). Then

$$\begin{aligned} \mathrm{E}[r_1] - R &= \frac{F_1 + F_1}{T} \mathrm{Cov}[r_1, r_m] && (14) \\ &= \frac{\mathrm{Cov}[r_1, r_m]}{\mathrm{V}[r_m]} \ \{ \ \mathrm{E}[r_m] - R \ \} && (15) \end{aligned}$$

where to get the second line I have used (13). This gives us the equilibrium excess rate of return (risk premium) on firm 1's shares or the first risky asset. (That on the other follows similarly.) The two lines interpret it in two different ways.

The first line (equation (14)) expresses this risk premium in terms of the market price of risk and the *covariance* between the rate of return on this asset and that on the risky market as a whole. This covariance is called the systematic or undiversifiable part of the risk of asset 1. The part that is uncorrelated with the market risk is the idiosyncratic or diversifiable part. Individuals can avoid it by diversifying their portfolio, so the market won't pay them for bearing that risk. You get a risk premium only for holding the risk of the market as a whole, which someone must hold.

The second line (equation (15)) relates the risk premium on asset 1 with that on the market as a whole. The factor of proportionality, $\text{Cov}[r_1, r_m]/\text{V}[r_m]$, tells us how well or poorly the risk of this asset is correlated with that of the market as a whole. It is called the *beta* of this asset, written β_1. If this is positive, then the risk on this asset is positively related to that of the market as a whole. By holding it, an investor is partaking to some extent in the risk of the market as a whole, and gets a risk premium for doing so. If the beta of this asset is negative, then on average this asset pays well when the market pays poorly, and vice versa. Therefore this asset is good for hedging, and investors are especially attracted to it for that reason. Then its risk premium is negative.

We can also find an expression for the value of a firm. Substituting from the relations between rates of return and totals (6) into the market-clearing condition (12) for firm 1, we get

$$T \left(\frac{\text{E}[\Pi_1]}{F_1} - R \right) = \frac{\text{V}[\Pi_1]}{F_1} + \frac{\text{Cov}[\Pi_1, \Pi_2]}{F_1}$$

or

$$T \, (\, \text{E}[\Pi_1] - R \, F_1 \,) = \text{V}[\Pi_1] + \text{Cov}[\Pi_1, \Pi_2] = \text{Cov}[\Pi_1, \Pi_1 + \Pi_2] \, .$$

Then

$$F_1 = \frac{\text{E}[\Pi_1] - \frac{1}{T} \, \text{Cov}[\Pi_1, \Pi_1 + \Pi_2]}{R} \tag{16}$$

Remember that R is the gross rate of return on the safe asset, that is, one plus the safe interest rate. Therefore the R in the denominator on the right hand side means that F_1 is a discounted present value. Of what? The firm's expected profits, minus a correction for risk. This correction depends inversely on the market's risk tolerance, and on the covariance between this firm's profits and the profits of the whole market, that is, only the systematic component of the risk.

4 General Theory of Contingent Claim Markets

Here we offer you the first step on the path to becoming true "rocket scientists" of finance, that is, to seven-figure as opposed to six-figure incomes.

4.1 Scenarios

When the participants in an economy face (and trade) risk, they do not know which of the possible outcomes will actually obtain. The analysis of this situation should begin by listing all possibilities. Each of these is called a "state of the world" in the technical jargon of general equilibrium theory, or a "scenario" in financial analysis.

In principle, each scenario should be a fully detailed description of one outcome, specifying everything that matters to anyone in the economy. In other words, once one particular scenario has transpired, there should be no more remaining uncertainty. Thus one scenario might be: "Friday the 13th, 8 a.m. The temperature is 41 degrees. X's car refuses to start and Y is coming down with a cold. ... " In practice, the participants in the economy and the economist analyzing the situation alike have to make do with much less detailed specification; it is a matter of judgment whether it is adequate for the context. For our purpose of introducing the ideas, we will think in terms of very simple examples – rain or shine, accident or no accident, and so on. At more advanced levels of financial rocket science, it is possible to recognize a whole continuum of different scenarios. In the capital asset pricing model of the previous section, one can in principle call each outcome corresponding to all possible levels of the two firms' profits (Π_1, Π_2) a different scenario, so we have a two-dimensional continuum of scenarios. Luckily that example could be (and was, above) analyzed using simpler methods.

Of course our interest focuses on the choices of individuals and the equilibria of their interaction in the face of uncertainty, that is, *before* they know which one of all the possible scenarios will actually transpire. For the present, we will suppose that the uncertainty is the same for all participants, that is, there is no asymmetric information about it. Asymmetric information can arise in different ways. (1) Different people may have different information about which among all possible scenarios may transpire. Thus one person may know that future interest rates are sure to be between 4 and 6 percent, and another may know that they are sure to be between 5 and 7 percent. They can both be right – there is an overlap in their information where the true number lies – but the basis for trade between them is problematic. (2) Some people may have better knowledge of the probabilities with which the different scenarios can arise; for example, they may know that some intermediate event has happened, and can update or condition the probabilities on this information. (This needs to be distinguished from merely having different subjective assessments of the probabilities.) (3) Each person has private knowledge of his own skills, preferences etc. while others don't know him so well. This results in adverse selection. (4) Some people can affect probabilities of outcomes by taking some action that others cannot observe. This creates moral hazard. Asymmetric information creates extra problems because individual choices become part of a game. For example, a potential buyer of a risky asset may ask himself "Why is the seller offering it to me at this seemingly attractive price? He may knows something horribly wrong with it." That can lead to a Groucho Marx like decision "I don't buy any asset that someone is willing to sell to me." We will study some aspects of the economics of asymmetric information later; its implications for finance must await another course.

4.2 Choice and Equilibrium

When people think about the possible future scenarios, they will be concerned with their consumption possibilities in each, and have different preferences about these in different scenarios. For example, the tradeoff between umbrellas and sunglasses will be quite different in the scenario "rain" than in the scenario "shine". People can also make trades across scenarios. For example, before anyone knows who will win the world series, two people can bet on this event. The bet is a contract or a promise to make certain transfers of money after the uncertainty is resolved. Note that once this has happened and one of the possible scenarios has actually happened, the loser pays the winner – the trade is unbalanced after the fact. But in equilibrium, trade must be balanced before the fact – for each person who takes one side of a bet, there must be another who takes the other side, and the terms of the bet (odds) are a relative price that brings about the equilibrium. And the preferences of each are such that each regards the trade as desirable before the fact, otherwise they would not have engaged in it voluntarily. For example, a NY Yankees supporter and a Florida Marlins supporter might each overestimate the chances of their team winning and bet on it. Or they might be realistic, and bet against the team they support, so that in the event the team loses, they at least have the consolation of winning some money; this is "hedging" behavior.

In the situation where uncertainty exists, and people are making trades across scenarios, where do they get the budget for this? They may have some wealth accumulated in the past or income from current activities, but they can also sell promises to deliver something in some scenarios and use the proceeds to buy other people's promises to deliver other things in other scenarios. Remember that a scenario is a complete description of everything that matters, so once the contract specifies the condition (scenario) in which that promise will have to be fulfilled, there is no doubt about the person's ability to deliver. In this analysis we assume that the law can enforce these contracts. The person cannot refuse to deliver if the occasion arises. Therefore these contracts are credible, can be treated just like any other goods or services that are bought or sold.

To simplify the exposition, let us leave out of our consideration everything except the trading of wealth across scenarios. Thus each person cares about how much wealth he will have available in the different scenarios. Of course he cares not for wealth as such but what goods and services it will enable him to consume; we are keeping in some background (leaving out of our explicit analysis) this step of converting wealth into consumption quantities in each scenario. We are also leaving out of consideration the trades between wealth or consumption right now and wealth or consumption in various scenarios of the future, that is, the subject of saving or investment under uncertainty. All that follows easily once the basic concepts are understood.

Label the scenarios 1, 2, ... Suppose that, before any trades in wealth (consumption ability) across scenarios, a person will have wealth amounts $\overline{W}_1$, $\overline{W}_2$, ... in the respective scenarios. Suppose one can write contracts of the following form: "I undertake to pay you \$1 if scenario i comes about, and nothing otherwise".

Such contracts, which promise a unit of purchasing power in one specified scenario, are called *Arrow-Debreu Securities,* after the economists Kenneth Arrow and Gerard Debreu who developed the concept. (Sometimes just "Arrow Securities" is used.) More generally, promises to deliver something in some scenario are called *contingent claims.* The "claim" part says that the person holding this promise (a piece of paper) is entitled to something, and the "contingent" part says that the entitlement is conditional or contingent upon some particular scenario or scenarios specified in advance in the contract actually happening. An Arrow-Debreu Security is a very simple or pure kind of contingent claim – the claim is to one monetary unit or unit of purchasing power, rather than to a particular good; the contingency is a single scenario. There can be more complicated contingent claims. For example, suppose we define a single or simple scenario to be each degree of temperature. Then a promise to deliver a pint of lemonade if the temperature at any time in the afternoon rises above 90 degrees, a mug of coffee if the temperature falls below 50 degrees, and nothing otherwise, is a complex contingent claim which promises different things in different combinations of the simple scenarios. Most financial assets are complicated contingent claims; for example a firm's debt is a claim to a specified stream of interest payments followed by a principal repayment in all those contingencies where the firm is able to meet these obligations, and default otherwise. Much of financial economics is about finding the prices of complicated contingent claims in terms of simpler ones.

An Arrow-Debreu Security is like a betting slip. You must buy it before the uncertainty is resolved (that is, before you know the outcome of the race). Suppose you have done this, and hold one scenario-1 Arrow-Debreu Security. When the uncertainty is resolved, either scenario-1 will transpire or it won't. If it does (the horse on which you had bet wins), you go to the issuer of your security (betting window), present your contract (betting slip), and collect your dollar. If some other scenario comes about (some other horse wins), your contract (betting slip) is worthless; you throw it away. This should make you think that what you would be willing to pay for such a contract at the initial time before uncertainty is resolved should have something to do with the probability of that scenario. That is correct, except that if people are not risk-neutral, their attitudes toward risk will affect the pricing. We will see more about this soon.

Suppose an individual wishes to consume more than his initial wealth $\overline{W}_1$ in scenario 1. For example, scenario 1 might be a situation where his house burns down, so $\overline{W}_1$ is low. He can consume $W_1 > \overline{W}_1$ by buying $(W_1 - \overline{W}_1)$ of the Arrow-Debreu Securities for scenario 1. To do this in a market economy, he will have to give up something else, and in our context he must sell Arrow-Debreu Securities for some other scenario or scenarios, say 2. Suppose he sells $(\overline{W}_2 - W_2)$ of thsee, so he will consumes $W_2 < \overline{W}_2$ if scenario 2 materializes. Let P_i denote the market price of the i-th Arrow-Debreu Security. His budget constraint is that the value of his purchases must not exceed what he has earned from his sales, that is,

$$P_1 \; (W_1 - \overline{W}_1) \leq P_2 \; (\overline{W}_2 - W_2) \, ,$$

or

$$P_1 \; W_1 + P_2 \; W_2 \leq P_1 \; \overline{W}_1 + P_2 \; \overline{W}_2 \, .$$

More generally, with several scenarios, this becomes

$$P_1 \; W_1 + P_2 \; W_2 + P_3 \; W_3 + \ldots \leq P_1 \; \overline{W}_1 + P_2 \; \overline{W}_2 + P_3 \; \overline{W}_3 + \ldots$$

I want to emphasize two points: [1] These markets are being held, the plans are made, and the prices are paid or received, all *before* the uncertainty is resolved. [2] Although the W_i are wealth amounts, in the budget constraint they are treated like quantities, because in this context they are the numbers of Arrow-Debreu securities being bought or sold. The prices are what it costs today to have the entitlement to a dollar if the specified scenario materalizes; their units will be like "so many cents on the dollar".

The economics of these markets for Arrow-Debreu securities (betting slips), and their mathematics, are exactly like the general equilibrium of exchange where 1 and 2 were ordinary goods. The next step is to bring in preferences. Most generally, people could have indifference maps in the space of $(W_1, W_2, \ldots)$, and we could represent these by utility functions, say $F(W_1, W_2, \ldots)$. In the context of uncertainty, a special and usual case is where preferences can be represented by an expected utility function

$$\pi_1 \; U(W_1) + \pi_2 \; U(W_2) + \ldots \, ,$$

where the π_i are the probabilities (objective or subjective) of the different scenarios. However the preferences are represented, the individual will choose $(W_1, W_2, \ldots)$ to maximize them.

Next we put together all the individuals. Each one, labelled h, will have initial wealth levels $(\overline{W}^h_1, \overline{W}^h_2, \ldots)$ in the different scenarios. Facing market prices $(P_1, P_2, \ldots)$ for the Arrow-Debreu Securities corresponding to all scenarios, his choice will produce demand functions (final wealth levels)

$$W^h_1(P_1, P_2, \ldots, \overline{W}^h_1, \overline{W}^h_2, \ldots),\ W^h_1(P_1, P_2, \ldots, \overline{W}^h_1, \overline{W}^h_2, \ldots),\ \ldots$$

In each scenario, the total desire of some individuals to sell Arrow-Debreu Securities for this scenario must equal the total desire of other individuals to buy Arrow-Debreu Securities for this scenario. That is, the total plans for final wealth must equal the total initial wealth. So in scenario 1, for example

$$\sum_h W^h_1(P_1, P_2, \ldots, \overline{W}^h_1, \overline{W}^h_2, \ldots) = \sum_h \overline{W}^h_1 .$$

We have a number of equilibrium conditions equal to the number of scenarios. Of these, one is redundant because of Walras' Law. The remaining ones determine the relative prices for the scenarios, $(P_1, P_2, \ldots)$.

If there is one risk-neutral trader, his objective function is

$$\pi_1\ W_1 + \pi_2\ W_2 + \ldots$$

If the prices $(P_1, P_2 \ldots)$ are not proportional to the probabilities $(\pi_1, \pi_2, \ldots)$, then this trader will take extreme positions (risks), long (holding large amounts of wealth) in only those scenarios where P_i/π_i is smallest, and short (negative amounts of wealth if permissible) in other scenarios. This is not compatible with equilibrium where the total amounts of wealth available in each state are fixed. Therefore equilibrium prices will have to be proportional to the probabilities. If no one is risk-neutral, then their risk-averse choices will generally keep the equilibrium prices of Arrow-Debreu Securities non-proportional to the probabilities of the scenarios. We will see and interpret examples of this shortly.

As usual, only relative prices matter. So in our model there is nothing to pin down the general level of the price vector $(P_1, P_2, \ldots)$. If there were some other commodity, then we could express these prices relative to it. For example, if there is money available right now, and it can be carried 1 for 1 into any one of the scenarios without any risk, then this "storage" activity is a perfect substitute for the composite contract that holds 1 unit of each of the Arrow-Debreu Securities corresponding to all scenarios, and therefore

$$P_1 + P_2 + \ldots = 1 .$$

If the uncertainty is about the future, and the rate of interest between now and the future is r, then the sum of prices of all the Arrow-Debreu Securities must equal the discounted present value of 1 sure future dollar, that is,

$$P_1 + P_2 + \ldots = 1/(1 + r) .$$

(Actually this is a sleight of hand. In this latest argument, people can transfer wealth between the now, when the Arrow-Debreu securities (ADS) are being traded, and the future, when uncertainty is resolved, one scenario materializes, and its ADS pays out. But the previous budget constraint required a balance between the values of a trader's endowment of ADS's (initial wealths $\overline{W}_i$) and consumption (final wealths W_i), with no transfer of wealth between now and the future (lending or borrowing). But the analysis can be done correctly with a full-fledged present value budget constraint. I have left this out only for simplicity of exposition.)

The mathematics of this general equilibrium is exactly the same as that of any other general equilibrium. And so are its properties. Most importantly, the equilibrium is Pareto efficient – it is impossible to increase the expected utility of any one individual without lowering that of another. Since indifference curves of expected utility tell us the trade-offs of wealth across scenarios, this means that the equilibrium achieves an efficient allocation of risk among these individuals.

We looked at an exchange equilibrium, but we can introduce firms and production without further complications. In other words, our previous study of general equilibrium has given, us as a by-product and without any further work, a complete theory of how markets help people cope efficiently with uncertainty.

4.3 Complete Markets

Of course, any theory is driven by its assumptions. In the above theory of Arrow-Debreu Securities we assumed away any problems of asymmetric information; we will pick these up later. But one other assumption slipped in unnoticed, namely that an Arrow-Debreu Security was available for each scenario. That was what enabled our traders to trade arbitrarily across scenarios. Of course they were subject only to the discipline of the market prices and budget constraints, but these things don't threaten efficiency of allocation in markets, in fact they promote it. If such a full set of Arrow-Debreu Securities is available, we say that markets are complete.

In reality, there are numerous financial instruments, but Arrow-Debreu Securities do not figure among them. Instead we have things like equity and debt. These offer more complex payoff patterns across scenarios. For example, suppose you hold one share of MicTel stock, and MicTel's dividends per share in different scenarios will be $(D_1, D_2, \ldots)$, then this is what your share will get you in these scenarios. (Remember that each scenario is a complete description, so there is no uncertainty about the magnitudes D_1, D_2 ... themselves; the only uncertainty is about which scenario will transpire, and therefore about which of these magnitudes you will actually receive.)

But there is a close connection between Arrow-Debreu Securities and such realistic financial instruments. Suppose a complete set of Arrow-Debreu Securities were available. Then your MicTel share would be fully equivalent to holding D_1 scenario-1 Arrow-Debreu Securities, D_2 scenario-2 Arrow-Debreu Securities, and so on – you will get exactly the same payoffs in each scenario from your MicTel stock as you would from this set of Arrow-Debreu Securities. Therefore the price of a MicTel share must equal

$$P_{\text{MicTel}} = P_1\ D_1 + P_2\ D_2 + \ldots$$

Your share cannot be worth any more; otherwise you would sell it and buy the equivalent set of Arrow-Debreu Securities, which is now a cheaper way of getting the same eventual consumption possibilities in all scenarios. And your share cannot be worth any less, otherwise someone else will buy it, sell the equivalent set of Arrow-Debreu Securities, and make a profit. These activities carry no risk because they offer more in *all* scenarios. (And there is no default risk because remember contracts are assumed to be fully credible.) Thus "riskless arbitrage" will keep the two sets of prices in line.

The idea generalizes immediately to any number of scenarios. Thus the simplest conceivable contingent claims, namely Arrow-Debreu Securities, provide the basis for finding the prices of all other contingent claims, no matter how complicated. (In cases where the set of scenarios is a continuum, the prices of the continuum of Arrow-Debreu Securities become the "pricing kernel".)

Now suppose there are two scenarios, and two firms, MicTel and BioWiz. MicTel's dividends per share in the two scenarios will be M_1 and M_2, and BioWiz's will be B_1 and B_2. (Note again that each scenario is a complete description of all relevant magnitudes.) Suppose there are no Arrow-Debreu Securities; the only available assets are shares in MicTel and BioWiz. Suppose I hold X_M MicTel shares and X_B BioWiz shares. My dividend receipts will be $X_M\ M_1 + X_B\ B_1$ in scenario 1 and $X_M\ M_2 + X_B\ B_2$ in scenario 2. Can I choose X_M and X_B so that these amounts are exactly what I would get if I could hold one scenario-1 Arrow-Debreu Security, that is, can I choose a portfolio that exactly *replicates* the payoff structure of this (hypothetical) security? That requires

$$\begin{aligned} X_M\ M_1 + X_B\ B_1 &= 1 \\ X_M\ M_2 + X_B\ B_2 &= 0 \end{aligned}$$

This pair of linear equations can be solved for X_M and X_B provided the determinant of coefficients is non-zero, that is, $M_1\, B_2 - B_1\, M_2 \neq 0$. This condition can be written in an economically more meaningful way, $M_1/B_1 \neq M_2/B_2$, which says that the two firms do not have the same payoff patterns across scenarios, that is, their payoff risks are not perfectly correlated.

Given this condition, the solution is

$$X_M = \frac{B_2}{M_1\, B_2 - B_1\, M_2}, \qquad X_B = \frac{-M_2}{M_1\, B_2 - B_1\, M_2}$$

Depending on the sign of the denominator, one or the other of these can be negative, that is, the portfolio that replicates a scenario-1 Arrow-Debreu Security can entail short sales of shares of one of the companies. But if financial markets allow short sales, then there is no problem about the replication. And an arbitrage argument exactly like the one that expressed the prices of complicated claims in terms of simple Arrow-Debreu Securities works the other way round too; the price of a scenario-1 Arrow-Debreu Security must be

$$P_1 = X_M \; P_{\text{MicTel}} + X_B \; P_{\text{BioWiz}} \, .$$

Similarly, another portfolio can replicate a scenario-2 Arrow-Debreu security, and its price can be found.

The analysis generalizes to arbitrary numbers of scenarios. So long as there are enough traded assets whose payoff structures across scenarios are sufficiently uncorrelated with each other, we can construct portfolios to replicate Arrow-Debreu Securities for all scenarios. (The precise condition using linear algebra is that the vectors of payoffs of available securities, depicted in the space whose dimension equals the number of scenarios, should *span* that space.) And once we have found portfolios equivalent to Arrow-Debreu Securities for all scenarios, we can price any other assets (contingent claims) of arbitrary complexity.

In other words, even if Arrow-Debreu Securities don't exist, they can be constructed, and therefore markets are complete, if there exists a sufficiently rich menu of other securities. In more realistic situations where uncertainty gradually unfolds over time, the same is true if reallocation of portfolios is possible at each period; this is called "dynamic hedging". And in that context, a much richer set of securities such as forward contracts and options is available for the purpose. If markets are not complete, we may still be able to calculate the "shadow prices" of hypothetical Arrow-Debreu securities.

This theory is the basis of the whole modern theory of derivative securities. For example, the Black-Scholes option pricing model works by observing that a combination of holding an option long and selling a suitable quantity of the underlying security short creates (replicates) a riskless asset, which can be priced in terms of the safe interest rate.

I expounded the theory in terms of two or three scenarios and their Arrow-Debreu securities. In real nontrivial applications, the random variables can take on values over a continuous range. Then there is a continuum of scenarios, and the prices of Arrow-Debreu securities over this continuum become functions. Then we have to do the above theory in "function spaces". But ultimately the combination of some securities to replicate others, and finding prices of one set of securities in terms of another set using no-arbitrage conditions, is just linear algebra. The details get complicated, but the basic idea is simple, and just what we have already learned. Take the trouble to master it, and you are on your way to designing new financial instruments. To summarize:

$$\text{Finance} = \text{General Equilibrium} + \text{Linear Algebra !}$$

4.4 Examples

Understanding of a general theoretical framework comes only from doing several examples that use it. So here are a few. All of them are based on a very simple story. Consider two corn farmers in an area isolated from others. Suppose there are local variations of climate, which affect the size of their crops. Together they consume the total output that is available in each climate scenario, but can trade promises where typically the farmer who has been lucky will give some output to the one who has been less fortunate. We run this example through variations of specifications and methods of analysis, starting with the simplest and gradually getting more complex.

4.4.1 Two-Scenario Geometric Analysis

First suppose there are two scenarios, in each of which exactly one of the farmers is lucky. The lucky one gets one ton of corn, and the unlucky one gets nothing. Both are equally averse to risk. Then there is an obvious efficient risk-sharing agreement: the lucky one should give half a ton to the unlucky one.

Figure 4 shows this in an Edgeworth box diagram. The axes show corn quantities, one for each scenario. The total output of corn is the same in either scenario, namely 1 ton; therefore the box is a square. The interpretation is that there is no aggregate risk in this mini-economy.

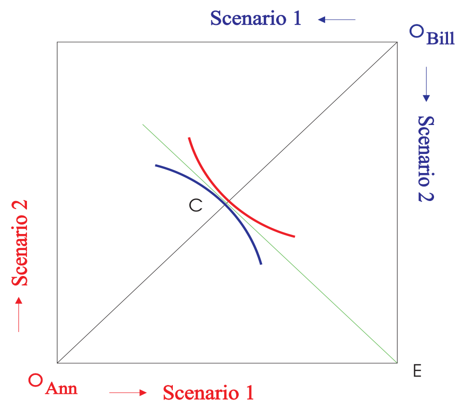

Figure 4: Perfect Insurance of Individual Risk

Each point in the box represents an allocation of the outputs between the two farmers, here named Ann and Bill. The southeast corner is the initial allocation or endowment point – Ann is the lucky one in scenario 1 and produces 1 ton; she is the unlucky one in scenario 2 and produces nothing. Bill's situation is the opposite. Thus there is individual risk. Any point in the box other than the southeast corner results from a trade across scenarios, where Ann agrees to give up some corn in scenario 1 in exchange for a promise by Bill to give up some in scenario 2.

For Ann, at any point on the 45-degree line from her origin O_A, the quantities of corn she consumes are equal in the two scenarios. Therefore this is her "line of no risk". Similarly the 45-degree line from O_B is Bill's line of no risk. Because the box is a square, the two lines coincide.

Write C_1^a and C_2^a for the quantities of Ann's corn consumption in the two scenarios, and write her expected utility function

$$EU^a = \tfrac{1}{2}\,U(C_1^a) + \tfrac{1}{2}\,U(C_2^a)\,.$$

The marginal rate of substitution is

$$\left.\frac{dC_2^a}{dC_1^a}\right|_{EU^a=\text{constant}} = \frac{\frac{1}{2}\,U'(C_2^a)}{\frac{1}{2}\,U'(C_1^a)}$$

Along her line of no risk, $C_1^a = C_2^a$, so the MRS reduces to the ratio of probabilities, $\frac{1}{2}/\frac{1}{2} = 1$. Similarly for Bill.

Now draw the line of slope negative 1 through the endowment point E. It cuts the line of no risk at the midpoint of the box. And we have just seen that both farmers' indifference curves have slope negative 1 at any point on the line of no risk. Therefore the midpoint of the box, corresponding to a mutual insurance arrangement to share output equally in both scenarios, is a competitive equilibrium of the risk markets. The slope of the line along which they exchange, that is, the relative price of the two Arrow-Debreu Securities is 1, the same as the ratio of the probabilities of the scenarios, $\frac{1}{2}/\frac{1}{2}$. Thus the purely individual risk that exists in this economy can be fully insured at prices that reflect just the probabilities; no risk premium arises.

Figure 5 illustrates some other aspects of risk-trading. Once again the two scenarios are equally likely, but the total amount of corn in the first scenario exceeds that in the second scenario. Thus there is aggregate risk, and scenario 2 is the "bad" one. The two farmers' lines of no risk are different. The endowment point E lies on Ann's line of no risk; she has equal outputs in the two scenarios. So she starts out with no risk; initially all of the aggregate risk falls on Bill.

For the same reasons as above, the MRS of each equals the ratio of probabilities of the scenarios, namely 1, at any point along his or her line of no risk. Thus Ann's MRS at E is 1, but Bill's MRS at E must be flatter if it is to reach slope 1 at the point where it meets his line of no risk. Thus the two farmers' indifference

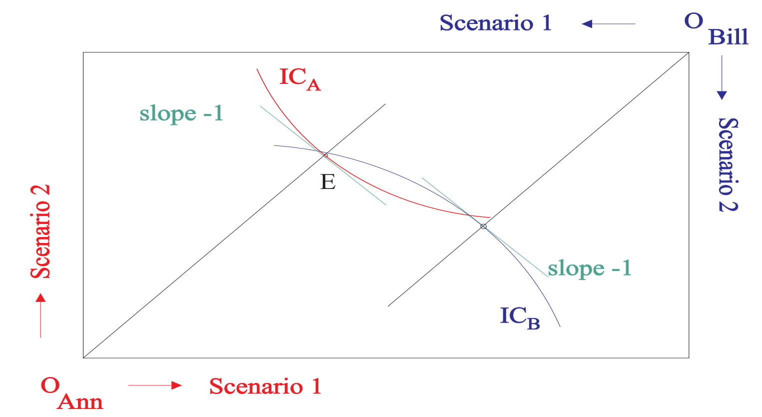

Figure 5: Shifting of Risk for a Price

curves intersect at E. Then there must be a lens-shaped region where they can both benefit from exchange. The line along which they exchange must be flatter than Ann's MRS at E, namely 1. In other words, the relative price of scenario-1 consumption must be less than the ratio of probabilities. Scenario 2 is the bad one where there is less total output available. Ann is giving up some of her scenario-2 output, and she does so only if she gets a price premium.

Remember that we are leaving out matters of great practical importance, arising from asymmetry of information. If the probability with which one farmer gets good weather, or the size of his crop, are not known in advance, then each farmer may pretend to be in a better situation in order to bargain for a more favorable deal ex ante. If a farmer's actual output depends on his effort as well as on the weather, then once an insurance contract has been signed, each has some incentive to shirk. And if the lucky farmer's actual output is not accurately observable ex post, then he has an incentive to hide some corn so as to give less to the other. Such problems can sometimes cause total failure of the risk market. Even if the market does not fail outright, the form of the contracts in it will be affected by the anticipation of these problems. We will study some theory of information asymmetry later in the course.

4.4.2 Two-Scenario Algebraic Example

Here we continue the story of Ann and Bill using algebra and calculus, which allow more generality than the pictures above. Suppose there are only two distinct scenarios, labelled 1 and 2, with probabilities π_1 and π_2 (with $\pi_1 + \pi_2 = 1$). In scenario 1, Ann gets output Y_1^a and Bill gets Y_1^b. In scenario 2, the respective quantities are Y_2^a and Y_2^b. For $i = 1$ and 2, write $Y_i = Y_i^a + Y_i^b$, the total output in scenario i.

They trade risk by trading Arrow-Debreu Securities. There are two of these, one corresponding to each scenario. A scenario-1 slip entitles the holder to one unit of corn if and when scenario 1 actually occurs, and nothing otherwise. Similarly for a scenario-2 security. A farmer who is to get some output in a particular scenario can sell the Arrow-Debreu Securities for that scenario.

When one scenario actually occurs, the person who has bought Arrow-Debreu Securities for that scenario will take them to the one who has sold them, and get the stated units of corn. No money changes hands at this time; the trade in promises to deliver occurred before the uncertainty was resolved, and all that is being done now is to fulfill those promises. The contracts that promise corn in the unrealized scenario are now worthless and are torn up.

Let P_1 and P_2 be the respective prices of the two Arrow-Debreu Securities in the risk-trading market that is held before anyone knows which scenario actually occurs.

Denote Ann's consumption quantities in the two scenarios by C_1^a and C_2^a. Her budget constraint, as explained previously, is

$$P_1\ C_1^a + P_2\ C_2^a = P_1\ Y_1^a + P_2\ Y_2^a \tag{17}$$

Similarly for Bill. Suppose each farmer has the von Neumann-Morgenstern utility function of consumption given by $\ln(C)$. This has the coefficient of relative risk aversion equal to 1. So Ann maximizes the expected utility function

$$\pi_1\ \ln(C_1^a) + \pi_2\ \ln(C_2^a)\,. \tag{18}$$

Similarly for Bill.

Ann's problem of maximizing (18) subject to (17) is exactly like the standard Cobb-Douglas consumer choice problem. Also conveniently, the coefficients π_1 and π_2, being the probabilities of the two scenarios, sum to 1. Therefore it leads to the demand function

$$C_i^a = \pi_i\ \frac{P_1\ Y_1^a + P_2\ Y_2^a}{P_i} \tag{19}$$

Bill's choice yields similar demand functions.

Now consider equilibrium in the market for scenario-1 Arrow-Debreu Securities. The total number of these sold by one farmer must equal that bought by the other farmer. This condition can be written as

$$C_1^a + C_1^b = Y_1^a + Y_1^b \equiv Y_1$$

which can also be interpreted as saying that total consumption in this scenario must equal total output in the same scenario: risk-trading markets cannot magically create output.

Adding up the two farmers' demands for scenario-1 Arrow-Debreu Securities and substituting into the equilibrium condition, we have

$$P_1\ (Y_1^a + Y_1^b) = \pi_1\ [\,(P_1\ Y_1^a + P_2\ Y_2^a) + (P_1\ Y_1^b + P_2\ Y_2^b)\,]$$

or

$$P_1\ Y_1 = \pi_1\ [\ P_1\ Y_1 + P_2\ Y_2\,]$$

Collecting the P_1 terms together, this simplifies to

$$\frac{P_1}{P_2} = \frac{\pi_1}{\pi_2}\ \frac{Y_2}{Y_1} \tag{20}$$

This gives us the relative price of the two Arrow-Debreu Securities in the risk-trading equilibrium, and we found that using only the equilibrium condition for scenario-1 securities. We have a confirmation, in this example, of the general properties that only relative prices can be determined, and one equilibrium condition is redundant by Walras' Law.

Observe some further properties of the solution: [1] Scenario-1 Arrow-Debreu Securities are relatively more valuable (P_1/P_2 is higher) if this scenario is relatively more likely (π_1/π_2 is higher). This is because both farmers are more willing to risk a low consumption in the relatively unlikely scenario 2, and the relative price of scenario-2 securities has to be lower to create a substitution effect in demands and equilibrate them with the supplies. [2] Scenario-1 securities are relatively less valuable if that scenario has more total output. In this situation there is aggregate risk, 2 being the "bad" scenario. The price of the entitlement to consume in that scenario has to be higher to reduce the demand by enough to ration the available number of promises to deliver output. [3] The division of the output risk between the two (who produces how much of Y_1 and how much of Y_2) does not matter for the relative price: individual or idiosyncratic risk can be insured fairly without the need for a price premium.

Dividing Ann's demand functions (19) in the two scenarios, and then using the equilibrium condition (20), we get

$$\frac{C_1^a}{C_2^a} = \frac{\pi_1}{\pi_2}\ \frac{P_2}{P_1} = \frac{Y_1}{Y_2}$$

Similarly for Bill. In other words, the two farmers bear the aggregate risk equally. This is because they are equally averse to risk. The next example shows the effects of different degrees of risk aversion, as well as some other additional considerations.

4.4.3 A Four-Scenario Example

Once again we suppose there are two farmers. Their von Neumann-Morgenstern utility functions are now different. One farmer, Cora, has COnstant (relative) Risk Aversion:

$$U(C) = \frac{1}{1-\rho}\ C^{1-\rho}$$

We will consider the effects of letting the coefficient of relative risk aversion ρ to take on different values. The other farmer, Ira, has Infinite Risk Aversion. His vNM utility function can be thought of as the limit of the above $U(C)$ as $\rho \to \infty$.

The output of each farmer can be either 1 or 2 with equal probability, and the risks of the two are independent. Thus there are four distinct scenarios with probability $\frac{1}{4}$ each: g is the "good state" where both have the higher output and total output equals 4; c where Cora has output 2 and Ira has output 1; i the other way round; and b is the "bad state" where both have the lower output and the total output is 2. In the risk market, therefore, four Arrow-Debreu Securities will be traded. Let P_g, P_c, P_i and P_b denote their prices. Write C_j^c and C_j^i for the consumption of the two in scenario j for $j = g,\ c,\ i,\ b$. Then Cora's budget constraint is

$$P_g\ C_g^c + P_c\ C_c^c + P_i\ C_i^c + P_b\ C_b^c = 2\ P_g + 2\ P_c + P_i + P_b$$

Subject to this, Cora maximizes the expected utility

$$\frac{1}{4}\ \frac{1}{1-\rho}\ \left[\,(C_g)^{1-\rho} + (C_c)^{1-\rho} + (C_i)^{1-\rho} + (C_b)^{1-\rho}\,\right]$$

(Exercise: In the spirit of "no pain, no gain", solve this problem and find Cora's demand functions.) Ira's budget constraint is

$$P_g\ C_g^i + P_c\ C_c^i + P_i\ C_i^i + P_b\ C_b^i = 2\ P_g + P_c + 2\ P_i + P_b$$

Ira's expected utility function (the limit as $\rho \to \infty$), can be shown to be equivalent to

$$\min[C_g^i, C_c^i, C_i^i, C_b^i]$$

The practical result is that Ira insists on perfect hedging, or equal consumption in all four scenarios. Therefore his demand functions are

$$C_g^i = C_c^i = C_i^i = C_b^i = \frac{2\ P_g + P_c + 2\ P_i + P_b}{P_g + P_c + P_i + P_b}$$

The equilibrium conditions are that the total demands must equal the total outputs in each scenario:

$$C_g^c + C_g^i = 4,\ \ C_c^c + C_c^i = 3,\ \ C_i^c + C_i^i = 3,\ \ C_b^c + C_b^i = 2$$

We can find three relative prices using any three of these equations. Numerical solution is required, and Table 1 shows the results.[1]

Observe the following points about the equilibrium: [1] The prices of the Arrow-Debreu Securities for the two middle scenarios are the same. I have chosen the freedom to normalize prices to make one of them equal 1; thus all prices are relative to this numeraire, and are not measured in units of output. But the fact that the price of the Arrow-Debreu Security for the other middle scenario equals 1 is an important result. The two middle states have the same total output but one farmer does better in one and the other in the other. We see that such idiosyncratic risk does not affect the price of risk. [2] For any given ρ, Cora gets to consume the same amount in these two middle scenarios, regardless of whether her own outcome is bad and Ira's is good or the other way round. Individual or idiosyncratic risk can be perfectly insured in the market. [2] Aggregate risk carries a price. The price of the Arrow-Debreu Security that gets you the entitlement to a unit of output in the "good" scenario is < 1, and that for the "bad" scenario is > 1. This is to induce Cora

[1] I thank Gordon Dahl of the University of Rochester, who did the programming when he was the preceptor for this class a few years ago.

Table 1: Risk Pricing and Allocation

Cora's Risk–Aversion Coefficient ρ	Cora's Consumption Quantities in Scenarios				Ira's Consumption Quantities (all Scenarios)	Prices of Arrow-Debreu Securities in Scenarios			
	g	c	i	b		g	c	i	b
0.001	2.50	1.50	1.50	0.50	1.50	0.99	1.00	1.00	1.01
0.50	2.60	1.60	1.60	0.60	1.40	0.78	1.00	1.00	1.64
1.00	2.68	1.68	1.68	0.68	1.32	0.63	1.00	1.00	2.44
2.00	2.81	1.81	1.81	0.81	1.19	0.51	1.00	1.00	4.99
10.00	2.99	1.99	1.99	0.99	1.01	0.02	1.00	1.00	1013

to substitute her consumption away from the bad scenario and toward the good scenario, that is, to bear the risk of having to consume unequal amounts in the two scenarios. [3] Ira is infinitely risk averse and is willing to pay any price to ensure equal consumption in all scenarios. The price he actually has to pay depends on Cora's attitude to risk. If Cora is almost risk-neutral (low ρ), then she is willing to bear risk for a very low price. Therefore the prices of the Arrow-Debreu Securities in the four scenarios are almost equal, and Ira's consumption is almost equal to his average output namely 1.5. [4] As Cora becomes more risk-averse, Ira has to pay a higher price to induce her to accept the risk. As Cora's ρ becomes very high, she demands a high price to bear risk. Thus the price spread between the good and the bad scenarios becomes very large, and Ira has to accept a level of consumption almost equal to his low output in order to ensure constancy of his consumption across scenarios.

4.5 Further reading (optional, for the ambitious)

Complicated as it may seem at first encounter, the above material is merely the beginning of a large body of financial theory. First, the examples developed above can easily be generalized, exactly as the 2-by-2 Edgeworth box diagram analysis of general equilibrium could be done algebraically with any number of goods, factors, consumers, and firms. Also, instead of the monetary payoffs of Arrow-Debreu securities in various scenarios, we can allow claims to specific goods or services in different scenarios. We can allow the gradual unfolding of the uncertain future. The original references on these topics are still valuable:

Kenneth J. Arrow, 1952, "The role of securities in the optimal allocation of risk-bearing." Reprinted in his *Essays in the Theory of Risk-Bearing,* North-Holland, 1971.

Gerard Debreu, 1953, "Une economique de l'incertain." Reprinted in translation in his *Mathematical Economics,* Cambridge University Press, 1983.

A more recent exposition of the basic micro theory is in

Andreu Mas-Colell, Michael Whinston, and Jerry R. Green, 1995, *Microeconomic Theory*, Oxford University Press.

A more specialized finance textbook along these lines is

Chi-fu Huang and Robert H. Litzenberger, 1988, *Foundations for Financial Economics*, North-Holland.

At the research frontier, financial economics increasingly treats choice and equilibrium in situations of incomplete markets and asymmetric information. This work is too complex to be cited here, but for the truly ambitious among you, I mention what is probably the best paper in finance for the last 25 years:

Albert S. Kyle, 1985, "Continuous auctions and insider trading," *Econometrica,* 53, November.

For an informative and enjoyable account of the development of modern finance, read

Peter L. Bernstein, 1992, *Capital Ideas,* Free Press.