ECO 305 – Fall 2001

Precepts Week 2 – Questions

This week we will see how some basic economic ideas translate into mathematical models, and how the results are then interpreted economically. We will do the two questions below; the answers will be available at the end of the precept.

Question 1:

St. Anford University is contemplating a fundraising campaign. Its experts have estimated that to raise x billion dollars, it must spend $C(x)$ billion dollars on the campaign.

(a) On the basis of economic intuition, what are the reasonable properties of this cost function?

(b) Does the function

$$C(x) = \begin{cases} 0.1 \ \exp(10\,x - 11) & \text{for all } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

have these properties?

(c) The University wishes to maximize its net receipts from the campaign. Find its optimum choice of x, and the resulting net receipts.

Question 2:

The small tropical island of Terra Nova is ruled by the benevolent dictator Oiko Nomos. The islanders can harvest coconuts or catch fish. To harvest Q_c coconuts requires $L_c = (Q_c)^2$ units of labor, and to catch Q_f fish requires $L_f = (Q_f)^2$ of labor. There are 80 units of labor available.

(a) Find the constraint on the quantities of coconuts and fish that the islanders can produce.

(b) The quantities produced are then consumed by the islanders, so the consumption quantities are $C_c = Q_c$ and $C_f = Q_f$. Oiko Nomos wants to maximize the "utility" of consumption, given by

$$U(C_c, C_f) = C_c\, C_f\,.$$

Find the optimum production plan that achieves this. What is the resulting utility level?

(c) Improvements in seagoing vessels now permit Terra Nova to trade with neighboring islands. The price of fish in terms of coconuts is 0.5 . That is, for every 2 fish this island sells (exports) to neighboring islands, it can buy (import) 1 coconut. Write down the expression for the national product of Terra Nova measured in coconut units, if it produces Q_c of coconuts and Q_f of fish.

(d) What production plan will maximize this national product? What is the maximum national product?

(e) National product equals national income, which can then be spent on consuming coconuts and fish. What is the budget constraint on the consumption quantities C_c, C_f?

(f) What consumption quantities will maximize the utility given in (b) above? What is the resulting utility?

(g) Which good is exported, and in what quantity? Which good is imported, and in what quantity?

(h) Does Terra Nova gain from trade?

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Precepts Week 2 – Answers

Question 1:

(a) $C(0) = 0$ – the cost should be zero if the campaign is not run.
It is possible that

$$\lim_{x \to 0} C(x) > 0\,.$$

There may be a fixed cost of running even the smallest campaign.

For all $x > 0$, we expect $C'(x) > 0$ – positive marginal cost. (If it were cheaper to run a larger campaign, then the smaller campaign could just mimic that strategy, and throw away any extra money if necessary.)

Also $C''(x) > 0$ – increasing marginal cost. Successive dollars are more and more costly to raise, because the campaign will begin touching the softest donors first (Sunday of the 25th reunion?) and then move on to the tougher ones.

(b) As $x \to 0$, $C(x) \to 0.1 \exp(-11) = \$1670 > 0$ So there is a fixed cost, but it is very small. Standard differentiation (repeated below) shows the marginal cost to be positive and increasing.

(c) To max $F(x) = x - C(x)$, the FONC is $F'(x) = 1 - C'(x) = 1 - 0.1 \times 10 \exp(10x - 11) = 1 - \exp(10x - 11) = 0$,which gives the critical point $10x - 11 = \ln(1) = 0$ so $x = 1.1$. There $F''(x) = -C''(x) < 0$ so we have a local max, and $F(1.1) = 1.1 - 0.1 \exp(0) = 1.1 - 0.1 = 1$.

Must compare this with $x = 0$ (not running the campaign at all), when obviously $F(0) = 0$. So $x = 1$ gives the global optimum.

Points to note: (1) You should be able to recognize, and use, the right ECO 102 lingo such as "fixed cost", and "increasing marginal cost". (2) Don't forgot the second-order condition. Imagine what would happen to a fundraiser who got the SOC wrong and minimized the net receipts! (3) Here we have an additional problem. Since there is a fixed cost, there is a local max at $x = 0$, so we need to check it against the other local max and see which is the global max.

In problem set questions of this kind, you will lose points for forgetting such matters.

Question 2:

(a) $(Q_c)^2 + (Q_f)^2 \leq 80$. This is the production possibility frontier (PPF) for the economy. For later reference, denote the function on the left hand side by $G(Q_c, Q_f)$.

(b) The Lagrangian is $L = Q_c\,Q_f + \lambda\,[80 - (Q_c)^2 - (Q_f)^2]$. So the FONCs are

$$Q_f - 2\lambda Q_c = 0, \quad Q_c - 2\lambda Q_f = 0$$

Therefore $Q_c = Q_f$, and using the constraint, each $= \sqrt{40} \approx 6.33$. Then utility $= 40$.

(c) National product $= Q_c + 0.5\,Q_f$

(d) The Lagrangian is $L = Q_c + 0.5\,Q_f + \mu\,[80 - (Q_c)^2 - (Q_f)^2]$. So the FONCs are

$$1 - 2\mu Q_c = 0, \quad 0.5 - 2\mu Q_f = 0$$

Then $Q_f = 0.5\,Q_c$. Substituting in the constraint,

$$80 = (Q_c)^2 + (1/4)\,(Q_c)^2 = (5/4)\,(Q_c)^2, \quad \text{so } (Q_c)^2 = 64,\ Q_c = 8, \quad \text{and } Q_f = 4.$$

The resulting national product is 10.

(e) $C_c + 0.5\,C_f \leq 10$.

(f) The Lagrangian is $L = C_c\,C_f + \nu\,[10 - C_c - 0.5\,C_f]$. So the FONCs are

$$C_f - \nu = 0, \quad C_c - 0.5\,\nu = 0$$

Therefore $C_c = 0.5\,C_f$. Substituting in the constraint, $C_f = 10$ and then $C_c = 5$. The resulting utility is 50 .

(g)Terra Nova exports $8 - 5 = 3$ units of coconuts, and imports $10 - 4 = 6$ units of fish. (Useful check that can spot errors: each unit of fish imported costs $3/6 = 0.5$ coconuts, this fits with the relative price you are given.)

(h) Utility with trade $= 50 > 40 =$ utility without trade, so yes, Terra Nova does gain from trade.

The general economic intuition is as follows. The marginal rate of substitution in consumption is

$$\left.\frac{dQ_f}{dQ_c}\right|_{\text{utility constant}} = \frac{\partial U/\partial Q_c}{\partial U/\partial Q_f} = \frac{Q_f}{Q_c}\,;$$

and the marginal rate of transformation along the production possibility frontier is

$$\left.\frac{dQ_f}{dQ_c}\right|_{\text{PPF}} = \frac{\partial G/\partial Q_c}{\partial G/\partial Q_f} = \frac{2\,Q_c}{2\,Q_f} = \frac{Q_c}{Q_f}\,.$$

In part (a), without any trade, at the society's optimum these two must be equal, so $Q_c = Q_f$, and each marginal rate is 1. If Terra Nova is run as a market economy, this will be the relative price of fish in terms of coconuts in the absence of trade.

The other islands offer to trade with Terra Nova at a smaller relative price for fish. Terra Nova can take advantage of this by (1) substitution in production, switching to less fish (4 instead of 6.33) and more coconuts (8 instead of 6.33); (2) substitution in consumption, toward more fish and fewer coconuts, in this case 10 fish and 5 coconuts. Note there is also an income effect, because trade is making Terra Nova better off as a whole. This could end up in it consuming more of both fish and coconuts than it did before. But in our example the substitution effect is stronger so Terra Nova consumes fewer coconuts (5) after trade than it did before (6.33). But it gets to consume enough more fish (10 instead of 6.33) that it is still better off in utility terms, which is after all what counts. Try drawing pictures where indifference curves have less substitution to see when Terra Nova would consume more of both goods with trade than without.

Some points to note about converting econ concepts to math and vice versa: (1) Observe carefully how the prices and quantities are put together in obtaining expressions for the national product etc. You may not believe this, but people doing similar questions on problem sets and exams in the past have tried to add the quantities and said that $Q_f + Q_c$ was the national product. You can't add fish to coconuts any more than you can add apples and oranges. Both must be converted into comparable values by multiplying by prices. And some people even multiplied the quantity of fish by the price of coconuts! (2) Observe carefully the relationship between production, consumption, imports, and exports. What your citizens can consume equals what your own economy produces plus what you import from the rest of the world. It is obvious when said like that; the math should become equally routine. (3) It is instructive to write the budget constraint in (e) as $C_c + 0.5\,C_f \leq Q_c + 0.5\,Q_f$, or $0.5\,(C_f - Q_f) \leq Q_c - C_c$. The left hand side in this final form is the value of Terra Nova's fish imports measured in coconut units, and the right hand side is the value of its coconut exports, again measured in coconut units. (The price of each coconut in coconut units is of course 1.) Therefore the budget constraint is just the balance of trade constraint - so long as the rest of the world is not giving you gifts, your exports must cover the value of your imports. (Of course when we recognize multiple periods, a country can borrow or lend, and a similar constraint will then hold in terms of discounted present values.)