ECO 305 – Fall 2003
Precept Week 5 material

Questions

Dual Cost Functions

Which of the following are legitimate dual cost functions $C(w, r, Q)$? Where they are legitimate, find the production functions corresponding to them.

(a) $\sqrt{w} + r\,Q$
(b) $[\,(w)^2 + (r)^2\,]^{1/2}\ Q$
(c) $[\,(w)^{1/2} + (r)^{1/2}\,]^2\ Q^2$
(d) $(w)^{1/3}\ (r)^{2/3} + [\,(w)^{1/2} + (r)^{1/2}\,]^2\ Q$

Properties of Dual Profit Function

Write P for the price of the output Q, and w, r for the prices of the inputs L, K respectively. Let $Q = F(K, L)$ be the production function. The dual profit function is defined as

$$\Pi^*(p, w, r) = \max\{\ p\,Q - w\,L - r\,K \mid Q = F(K, L)\,\}$$

Some conditions on F are needed for this problem to have a solution; basically F must have diminishing returns to scale as well as to each factor alone. But we take these for granted here. We want to examine some properties of the function Π^*. Specifically, we want to show that Π^* is:

(1) Homogeneous degree one in (P, w, r) jointly
(2) Convex in these prices (we will do them one at a time but joint is true also)
(3) Hotelling's Lemma: the firm's optimum choices of the output supply and input demands are given by

$$Q = \partial\Pi^*/\partial P, \quad L = -\ \partial\Pi^*/\partial w, \quad K = -\ \partial\Pi^*/\partial r$$

Consumer Theory Review

A consumer maximizes utility $U(X, Y)$ subject to $P_x\,X + P_y\,Y = M$. It is observed that as P_x increases (holding P_y, M constant), his optimally chosen X increases. Now consider the effects of increasing P_y, holding P_x and M constant. Does Y increases or decrease? What about X?

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Answers

Dual Cost Functions

Which of the following are legitimate dual cost functions $C(w, r, Q)$?

(a) $\sqrt{w} + r\,Q$

Not homogeneous of degree 1 in (w, r)

(b) $[\,(w)^2 + (r)^2\,]^{1/2}\ Q$

Not concave in (w, r). (The contours of cost in (w, r) space for fixed Q are just the quarter-circles $(w)^2 + (r)^2$ = constant. But a concave function would have contours of the opposite curvature, as one can see by taking a horizontal slice of any of the concave production functions that appeared in the graphics handout of October 9.)

(c) $[\,(w)^{1/2} + (r)^{1/2}\,]^2\ Q^2$

This is legitimate, in fact a special case of the CES family. Concavity can also be seen by noting that

$$[\,(w)^{1/2} + (r)^{1/2}\,]^2 = w + r + 2\,\sqrt{w\,r}$$

Note there is no need for cost to be homogeneous of any degree in Q; the power of Q being greater than 1 signifies that there are increasing marginal costs, corresponding to diminishing marginal returns to scale in production.

Now the input demands are found by Hotelling's Lemma:

$$\begin{aligned} L &= 2\,[\,(w)^{1/2} + (r)^{1/2}\,]\,\tfrac{1}{2}\,w^{-1/2}\ Q^2 = [\,1 + (r/w)^{1/2}\,]\ Q^2 \\ K &= 2\,[\,(w)^{1/2} + (r)^{1/2}\,]\,\tfrac{1}{2}\,r^{-1/2}\ Q^2 = [\,(w/r)^{1/2} + 1\,]\ Q^2 \end{aligned}$$

Note that input prices enter into each function only as the ratio w/r; this is as it should be since each optimal input quantity must be homogeneous of degree 0 in the input prices. Now we can eliminate w/r between these equations to get the production function

$$1 = \left(\frac{r}{w}\right)^{1/2}\ \left(\frac{w}{r}\right)^{1/2} = \left(\frac{L}{Q^2} - 1\right)\ \left(\frac{K}{Q^2} - 1\right)$$

so

$$0 = \frac{L\,K}{Q^4} - \frac{L}{Q^2} - \frac{K}{Q^2}$$

which yields the solution

$$Q = \left[\ L^{-1} + K^{-1}\ \right]^{-1/2}$$

This is exactly like the "duality" of utility and expenditure functions we saw in Problem Set 3 Question 1 part (a).

(d) $(w)^{1/3}\ (r)^{2/3} + [\,(w)^{1/2} + (r)^{1/2}\,]^2\ Q$

This is legitimate. The first term is the sunk cost; the production process comprising this part is Cobb-Douglas, proportional to $L^{1/3} K^{2/3}$. The second term is the variable cost; it being proportional to Q, the AVC is constant and equal to the MC. This part of the production process is similar to that in part (c) above except for the returns to scale. The production function is

$$Q = \left[\, L^{-1} + K^{-1} \, \right]^{-1}$$

Derivation is left for you as an exercise.

Properties of Dual Profit Function

Homogeneity is obvious: For any scaling factor s,

$$sp\,Q - sw\,L - sr\,K = s\,[p\,Q - w\,L - r\,K]\,.$$

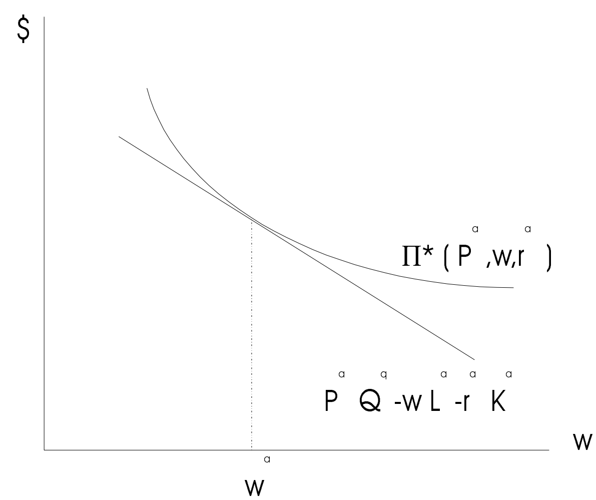
Therefore s does not affect the optimal choices of K, L and Q. All it does is to scale up the amount of the resulting profit by s. That is homgeneity of degree 1.

Convexity and Hotelling's Lemma are proved together. We show the slightly unusual case of an input demand.

Take an initial situation labelled a with prices (P^a, w^a, r^a) and quantities (Q^a, L^a, K^a). Now change w. The old quantities are still feasible and yield profit $P^a\,Q^a - w\,L^a - r^a\,K^a$. The new maximum profit when quantities are adjusted optimally can be no less, so

$$\Pi^*(P^a, w, r^a) \geq P^a\,Q^a - w\,L^a - r^a\,K^a$$

for all w, and the two are equal when $w = w^a$.



In the figure, the RHS of the inequality is a line of slope $-L^a$. This line is tangential to the profit function graph, so

$$L^a = -\partial\Pi^*/\partial w|_a$$

And the profit function graph lies entirely above its tangent at a. But a could be any point. Therefore the profit function is convex as a function of w (it is decreasing but convex - first derivative negative and second derivative positive) and its derivative with respect to w at any point is minus of the optimal labor demand at that point.

Consumer Theory Review

Since X increases as P_x increases, X is a Giffen good. Therefore it is an inferior good. Since there are only two goods, the other one Y cannot also be inferior. Therefore it cannot be Giffen. Therefore as P_y increases, Y must decrease.

Next, since there are only two goods, they must be substitutes. Therefore an increase in P_y generates a substitution effect in favor of X. Also, the income effect of an increase in P_y is like a reduction in income, and X is an inferior good. Therefore the income effect of an increase in P_y leads to an increase in X. With both the effects acting in the same direction, the quantity of X unambiguously increases.

We can see this using the mathematics of the Slutsky equations. We have

$$\left.\frac{\partial X}{\partial P_x}\right|_{u=\text{const}} = \left.\frac{\partial X}{\partial P_x}\right|_{M=\text{const}} + X\,\frac{\partial X}{\partial M}$$

We know that the LHS is ≤ 0 always, and we are told that here the first term on the RHS is positive. Therefore the second term must be negative: $\partial X/\partial M < 0$.

Next, from the budget constraint

$$P_x\,\frac{\partial X}{\partial M} + P_y\,\frac{\partial Y}{\partial M} = 1$$

and we just saw that $\partial X/\partial M < 0$, so $\partial Y/\partial M > 0$. Also, in

$$\left.\frac{\partial Y}{\partial P_y}\right|_{u=\text{const}} = \left.\frac{\partial Y}{\partial P_y}\right|_{M=\text{const}} + Y\,\frac{\partial Y}{\partial M}$$

the LHS is ≤ 0. Then the first term on the RHS must be < 0.

Finally, along $U(x,y) = \text{constant}$, we have $dy/dx < 0$. But $\partial Y/\partial P_y|_{u=\text{const}} \leq 0$. Therefore $\partial X/\partial P_y|_{u=\text{const}} \geq 0$. Use all this in the Slutsky equation

$$\left.\frac{\partial X}{\partial P_y}\right|_{u=\text{const}} = \left.\frac{\partial X}{\partial P_y}\right|_{M=\text{const}} + Y\,\frac{\partial X}{\partial M}$$

The LHS is ≥ 0, and the second term on the RHS is < 0, so the first term on the RHS must be > 0.