

## Questions

Here we will study some properties of a demand system that is often used in oligopoly theory, and then make a start toward that theory. The new conceptual issues are in Question 1. The other two questions are mostly mechanical computations. Try to do one yourself, and then read and keep the rest for reference. Please send me an e-mail if you find typos.

### Question 1 – Demand Functions

There are three goods, labelled 0, 1, and 2. The quantities are denoted by  $x_0$ ,  $x_1$ , and  $x_2$ . All prices are measured in units of good 0, so the prices are denoted by 1,  $p_1$ , and  $p_2$ .

The utility function is

$$u(x_0, x_1, x_2) = x_0 + a_1 x_1 + a_2 x_2 - \frac{1}{2} [b_1 (x_1)^2 + 2k x_1 x_2 + b_2 (x_2)^2]$$

and the budget constraint is

$$x_0 + p_1 x_1 + p_2 x_2 = I,$$

(a) What are the first-order necessary conditions (FONCs) for the maximization? (You will find it easier to substitute out  $x_0$  than to do Lagrange.)

(b) What are the second-order necessary conditions (SONCs) and sufficient conditions (SOSCs) for the maximization? (Here for once we need the full set of conditions you learned in your math courses.)

(c) Solve the FONCs to express each of  $p_1$  and  $p_2$  as a function of  $(x_1, x_2)$ . These are the “inverse demand functions”.

(d) Solve the FONCs to express each of  $x_1$  and  $x_2$  as a function of  $(p_1, p_2)$ . These are the “direct demand functions”. Under what conditions on  $(a_1, a_2, b_1, b_2, k)$  are the two goods 1 and 2 substitutes? When are they complements? And should we be using the Hicksian definition of substitutes/complements or the Marshallian one?

(e) Take the function linking  $p_1$  to  $(x_1, x_2)$  in (c). Holding  $x_2$  fixed, calculate the slope of the inverse demand curve in numerical value, namely  $-\partial p_1 / \partial x_1$ . Then take function linking  $x_1$  to  $(p_1, p_2)$  in (d). Holding  $p_2$  fixed, calculate the numerical value of the slope of the demand curve as it would look in the conventional diagram with  $p_1$  on the vertical axis and  $x_1$  on the horizontal axis. Which of the two slopes you calculated is larger in numerical value? Can you think of an economic intuition for this?

### Question 2 – Quantity-Setting (Cournot) Duopoly

Now suppose each of the goods 1 and 2 is produced by a single firm. Each firm chooses its quantity. Each has a constant per unit (average = marginal) cost,  $c_1$  for firm 1 and  $c_2$  for firm 2. The profit of firm 1 is

$$\Pi_1 = (p_1 - c_1) x_1.$$

(a) Using the inverse demand function in Question 1(c), express this as a function of  $(x_1, x_2)$ .

(b) If firm 1 thinks that firm 2 is choosing some particular quantity  $x_2$ , what is the condition for the choice of  $x_1$  to maximize firm 1's profit?

(c) If firm 2 likewise thinks that firm 1 is choosing some particular quantity  $x_1$ , what is the condition for the choice of  $x_2$  to maximize firm 2's profit?

(d) If each firm chooses its own quantity to maximize its own profit, regarding the quantity of the other as fixed, then the outcome is called the Cournot equilibrium of this duopoly. Take the above conditions for the two firms' maximization, and solve the two equations simultaneously to obtain the quantities in Cournot equilibrium in terms of the given algebraic constants (parameters) of the problem:  $(a_1, a_2, b_1, b_2, k, c_1, c_2)$ .

### Question 3 – Price-Setting (Bertrand) Duopoly

Again suppose each of the goods 1 and 2 is produced by a single firm. But now each firm chooses its price. Each has a constant per unit (average = marginal) cost,  $c_1$  for firm 1 and  $c_2$  for firm 2. Write the direct demand function as

$$x_1 = \alpha_1 - \beta_1 p_1 + \kappa p_2, \quad x_2 = \alpha_2 + \kappa p_1 - \beta_2 p_2.$$

Firm 1's profit is

$$\Pi_1 = (p_1 - c_1) x_1 = (p_1 - c_1) (\alpha_1 - \beta_1 p_1 + \kappa p_2).$$

(a) If firm 1 thinks that firm 2 is choosing some particular price  $p_2$ , what is the condition for the choice of  $p_1$  to maximize firm 1's profit?

(b) If firm 2 likewise thinks that firm 1 is choosing some particular price  $p_1$ , what is the condition for the choice of  $p_2$  to maximize firm 2's profit?

(c) If each firm chooses its own price to maximize its own profit, regarding the price of the other as fixed, then the outcome is called the Bertrand equilibrium of this duopoly. Take the above conditions for the two firms' maximization, and solve the two equations simultaneously to obtain the prices in Bertrand equilibrium in terms of the given algebraic constants (parameters) of the problem:  $(\alpha_1, \alpha_2, \beta_1, \beta_2, k, c_1, c_2)$ .

## Answers

### Question 1 – Demand Functions

(a) Solve for  $x_0$  from the budget constraint and substitute in the utility function to write it as a function of  $x_1$  and  $x_2$  alone:

$$u(x_1, x_2) = I - p_1 x_1 - p_2 x_2 + a_1 x_1 + a_2 x_2 - \frac{1}{2} [b_1 (x_1)^2 + 2k x_1 x_2 + b_2 (x_2)^2]$$

The FONCs for maximizing this are

$$\begin{aligned} -p_1 + a_1 - b_1 x_1 - k x_2 &= 0 \\ -p_2 + a_2 - k x_1 - b_2 x_2 &= 0 \end{aligned}$$

(b) The SONCs are that the matrix of the second-order partial derivatives of  $u$ , namely

$$\begin{pmatrix} -b_1 & -k \\ -k & -b_2 \end{pmatrix}$$

is negative semi-definite, or

$$b_1 \geq 0, \quad b_2 \geq 0, \quad b_1 b_2 - k^2 \geq 0.$$

The SOSCs require the matrix to be negative definite, for which these same inequalities have to be strict.

(c) The inverse demand functions follow immediately from the FONCs:

$$p_1 = a_1 - b_1 x_1 - k x_2, \quad p_2 = a_2 - k x_1 - b_2 x_2$$

(d) The simplest way to get the direct demand functions is to write the inverse ones in vector-matrix form:

$$\begin{pmatrix} b_1 & k \\ k & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 - p_1 \\ a_2 - p_2 \end{pmatrix}$$

Then

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} b_1 & k \\ k & b_2 \end{pmatrix}^{-1} \begin{pmatrix} a_1 - p_1 \\ a_2 - p_2 \end{pmatrix} \\ &= \frac{1}{b_1 b_2 - k^2} \begin{pmatrix} b_2 & -k \\ -k & b_1 \end{pmatrix} \begin{pmatrix} a_1 - p_1 \\ a_2 - p_2 \end{pmatrix} \\ &= \frac{1}{b_1 b_2 - k^2} \begin{pmatrix} (a_1 - p_1) b_2 - (a_2 - p_2) k \\ -k (a_1 - p_1) + b_1 (a_2 - p_2) \end{pmatrix} \end{aligned}$$

For these solutions to be properly defined, we need the SOS,  $b_1 b_2 - k^2 > 0$ . But if  $b_1 b_2 - k^2 = 0$ , we can interpret the right hand side as a limiting case where the price-derivatives go to infinity (perfect substitutes).

Goods 1 and 2 are substitutes if the cross-price derivatives are positive, which happens if  $k > 0$ , and complements if the cross-price derivatives are negative, that is  $k < 0$ . Utility being quasi-linear, there is no difference between the Hicksian and Marshallian concepts.

(e) For the inverse demand function, we have

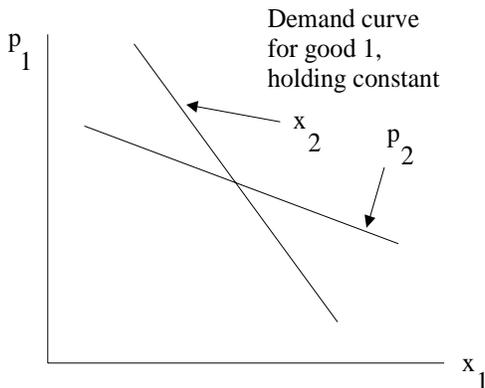
$$-\partial p_1 / \partial x_1 = b_1$$

For the direct demand function, holding  $p_2$  constant,

$$-\partial x_1 / \partial p_1 = b_2 / (b_1 b_2 - k^2)$$

The numerical value of the slope of the demand curve as conventionally drawn (with  $p_1$  on the vertical axis and  $x_1$  on the horizontal axis) is then just the reciprocal,  $(b_1 b_2 - k^2) / b_2$ . Now obviously  $b_1 b_2 - k^2 < b_1 b_2$ . Since  $b_2 > 0$ , dividing by it we have  $(b_1 b_2 - k^2) / b_2 < b_1$ . So the inverse demand curve for firm 1 is flatter when firm 2's price is held constant than it is when firm 2's quantity is held constant. And this is true regardless of whether  $k > 0$  or  $k < 0$ , that is, regardless of whether the goods are substitutes or complements.

To see the intuition, first consider the case of substitutes, say Coke and Pepsi. While the price of Pepsi is held constant, if more Coke is put on the market, the price that people are willing to pay for Coke will drop. The increased quantity will be taken up by some people switching from other drinks to Coke, and some by people switching from Pepsi to Coke. But now suppose Pepsi wants to hold its quantity constant, not its price. To hold its quantity constant (keep its customers), Pepsi must lower its price to some extent. That further lowers the price that people are willing to pay for Coke. Therefore the inverse demand curve for Coke is steeper (has a more negative slope) when the quantity of Pepsi is held constant than when the price of Pepsi is held constant. The figure below shows this schematically, hopefully making it easier to remember.



Next consider the case of complements, say wine and cheese. While the price of cheese is held constant, if more wine is put on the market, the price that people are willing to pay

for wine will drop. Some people will drink more wine and eat more cheese with it; others will switch from other goods (say beer and pretzels) toward the now cheaper wine-cheese combo. But now suppose the quantity of cheese is to be held constant, not the price. To do this, the new demand for cheese must be cut back, which requires raising the price of cheese. But then the wine-cheese combo becomes less attractive. People will be willing to buy the higher quantity of wine that is now on the market only if it becomes even cheaper. So again the inverse demand curve for wine is steeper (has a more negative slope) when the quantity of cheese is held constant than when the price of cheese is held constant.

This intuition is one of the trickier ideas in economics. But it is important. It says that if each of two firms thinks that the other is holding its quantity constant, it sees itself as facing a steeper (less elastic) demand curve, and therefore charges a higher price, than it would if it thought that the other was holding its price constant. So duopolistic price competition leads to lower prices, and is therefore more beneficial to consumers, than duopolistic quantity competition. This result is developed more formally in the oligopoly handout, p. 12. There the intuition is developed yet another way (using the direct demand functions), and given a name: the Le Chatelier - Samuelson Principle. Samuelson first developed the idea in economics, and named it after Le Chatelier's principle in thermodynamics.

## Question 2 – Quantity-Setting (Cournot) Duopoly

(a) Profit of firm 1 as function of  $(x_1, x_2)$ :

$$\begin{aligned}\Pi_1 &= [(a_1 - c_1) - b_1 x_1 - k x_2] x_1 \\ &= (a_1 - c_1 - k x_2) x_1 - b_1 (x_1)^2.\end{aligned}$$

(b) Therefore, when  $x_2$  is fixed and  $x_1$  is being chosen to maximize  $\Pi_1$ ,

$$\frac{\partial \Pi_1}{\partial x_1} = a_1 - c_1 - k x_2 - 2 b_1 x_1 = 0,$$

$$\frac{\partial^2 \Pi_1}{\partial x_1^2} = -2 b_1 < 0.$$

(c) Similarly for firm 2, the FONC is

$$a_2 - c_2 - k x_1 - 2 b_2 x_2 = 0$$

(d) The two simultaneous equations for the Cournot equilibrium quantities  $(x_1, x_2)$  can be written in vector-matrix form,

$$\begin{pmatrix} 2 b_1 & k \\ k & 2 b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 - c_1 \\ a_2 - c_2 \end{pmatrix}$$

Then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 b_1 & k \\ k & 2 b_2 \end{pmatrix}^{-1} \begin{pmatrix} a_1 - c_1 \\ a_2 - c_2 \end{pmatrix}$$

$$\begin{aligned}
&= \frac{1}{4b_1b_2 - k^2} \begin{pmatrix} 2b_2 & -k \\ -k & 2b_1 \end{pmatrix} \begin{pmatrix} a_1 - c_1 \\ a_2 - c_2 \end{pmatrix} \\
&= \frac{1}{4b_1b_2 - k^2} \begin{pmatrix} 2(a_1 - c_1)b_2 - (a_2 - c_2)k \\ -k(a_1 - c_1) + 2b_1(a_2 - c_2) \end{pmatrix}
\end{aligned}$$

If the two firms'  $(a_1 - c_1)$  and  $(a_2 - c_2)$  are sufficiently different, one of the quantities  $x_1$  and  $x_2$  can become negative. This cannot be an equilibrium; instead that firm produces zero and the other becomes effectively a monopolist. However, positive quantities for both firms in Cournot equilibrium are compatible with some cost or demand differences, even when the products are perfect substitutes. That could not happen in a perfectly competitive equilibrium with homogeneous products.

### Question 3 – Price-Setting (Bertrand) Duopoly

(a) Firm 1's profit as function of  $(p_1, p_2)$ :

$$\Pi_1 = (p_1 - c_1) (\alpha_1 - \beta_1 p_1 + \kappa p_2)$$

Therefore, holding  $p_2$  constant,

$$\begin{aligned}
\frac{\partial \Pi_1}{\partial p_1} &= (\alpha_1 - \beta_1 p_1 + \kappa p_2) - \beta_1 (p_1 - c_1) \\
&= (\alpha_1 + \beta_1 c_1) - 2\beta_1 p_1 + \kappa p_2
\end{aligned}$$

and

$$\frac{\partial^2 \Pi_1}{\partial p_1^2} = -2\beta_1 < 0$$

Therefore the FONC that defines firm 1's optimal choice of  $p_1$  given firm 2's price is

$$(\alpha_1 + \beta_1 c_1) - 2\beta_1 p_1 + \kappa p_2 = 0$$

(b) Similarly the FONC for firm 2 is

$$(\alpha_2 + \beta_2 c_2) - 2\beta_2 p_2 + \kappa p_1 = 0$$

(c) The FONCs can be written in vector-matrix notation

$$\begin{pmatrix} 2\beta_1 & -\kappa \\ -\kappa & 2\beta_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 c_1 \\ \alpha_2 + \beta_2 c_2 \end{pmatrix}$$

Then

$$\begin{aligned}
\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} &= \begin{pmatrix} 2\beta_1 & -\kappa \\ -\kappa & 2\beta_2 \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 + \beta_1 c_1 \\ \alpha_2 + \beta_2 c_2 \end{pmatrix} \\
&= \frac{1}{4\beta_1\beta_2 - \kappa^2} \begin{pmatrix} 2\beta_2 & \kappa \\ \kappa & 2\beta_1 \end{pmatrix} \begin{pmatrix} \alpha_1 + \beta_1 c_1 \\ \alpha_2 + \beta_2 c_2 \end{pmatrix}
\end{aligned}$$

This can be simplified somewhat using the relationships between the parameters  $(\alpha_1, \alpha_2, \beta_1, \beta_2, \kappa)$  of the direct demand functions and the parameters  $(a_1, a_2, b_1, b_2, k)$  of the inverse demand functions.