ECO 305 – Fall 2003
Precepts Week 10

Question 1:

Suppose someone offers you the opportunity to play the following game. This person will toss a fair coin repeatedly until heads appear. If this happens on the very first toss, you get \$2. If on the second (that is, tails on the first toss is followed by heads on the second toss), you get \$4. And so on; if $(n-1)$ tails in succession are followed by heads on the nth toss, you get $\$\,2^n$.

(a) Calculate the mathematical expectation of your (random variable) winnings W.

(b) What sure sum would you be willing to pay in advance to be given the opportunity of participating in this game?

(c) Suppose your von Neumann-Morgenstern utility function is $\sqrt{W}$. Calculate your expected utility. What sure sum, if offered to you instead of the game, would give you the same utility?

(d) Suppose your von Neumann-Morgenstern utility function is $\ln W$. Calculate your expected utility. What sure sum, if offered to you instead of the game, would give you the same utility?

(e) Suppose your von Neumann-Morgenstern utility function is $-1/W$. Calculate your expected utility. What sure sum, if offered to you instead of the game, would give you the same utility?

(f) What is the coefficient of relative risk aversion for each of the utility functions in (c)-(e)?

Question 2:

Consider the following simplified version of "Who Wants To Be A Millionaire". You have reached the \$32,000 level. (Any Princeton student should be able to do that.) Now you face a succession of questions, each with two possible answers. If you answer a question correctly, whether because you know the correct answer or because you make a lucky guess, you proceed to the next higher prize level. We call your first question (the one that if correctly answered takes you to the \$64,000 level) "the \$64,000 question" for short; similar abbreviations apply to the following levels. The prize doubles at each level. If you answer the \$64,000 question correctly and so reach the \$64,000 level, you face the \$128,000 question, and so on to \$256,000, and \$512,000. If you reach the \$512,000 level, you face one final question, and the correct answer to it will win you \$1,024,000.

At any level, when you get the question that can take you to the next level, you may choose not to answer it, and leave with the prize of the level you have already reached. At any level, if you answer the next question and your answer is wrong, you will leave with only \$32,000.

At each level, before you have seen the question that can take you to the next level, there is a probability that you know the correct answer. The questions get successively harder, so these probabilities decline from one level to the next. Before you see the \$64,000 question, the probability that you know the answer is 40 percent. If you answer this question successfully to reach the \$64,000 level, and before you see the \$128,000 question, the probability that you know the answer to that is 35 percent. Similarly, the probabilities are 30 percent for the \$256,000 question, 25 percent for the \$512,000 question, and 20 percent for the \$1,024,000 question.

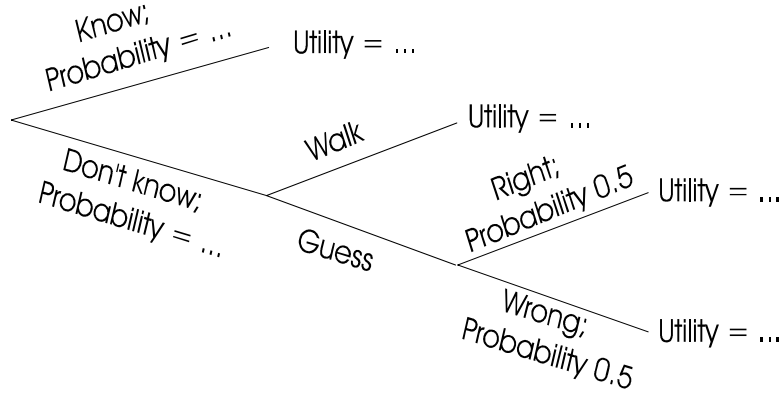
At each level, once you have seen the question, you will know for sure whether you know the correct answer. Thus there is no possibility that you are confident but wrong, or that you know the correct answer but fail to realize that you know.

At each level, after you see the question, if you know the correct answer, of course you will give it. If you don't, you have to decide whether to make a guess, which at any level has a 50 percent chance of being correct, or to walk, that is, leave with the amount of the level you have reached. Remember that if you choose to guess and are lucky, you proceed to the next level, but if you choose to guess and are unlucky, you will have to leave with only \$32,000, no matter what level you had reached.

Your von Neumann-Morgenstern utility function is $\log_2(W/32000)$, where W is the amount of dollars of your prize. (Note that logs are to base 2, not 10 or e.)

(a) What are the utility numbers corresponding to the various possible levels of prizes?

(b) Find your optimal strategy, namely your plan of action that prescribes, at each level, whether to make a guess if you don't know the answer to the question, or to walk with the prize of the level you have reached. You have to begin at the end (where you have already reached the $512,000 level) and work your way backward. At each stage you will find it useful to draw a mini "decision tree" like the one shown here, with the appropriate probability and utility values filled where ... is shown:



(c) What is your expected utility at the initial situation where you have reached the $32,000 level and are about to receive the $64,000 question? What amount of sure dollars would give you the same utility?

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Answer 1:

(a) The probability that heads show for the first time on the nth toss is 2^{-n}. Therefore your expected winnings are

$$\mathrm{E}[W] = \sum_{n=1}^{\infty} 2^{-n}\, 2^{n} = \sum_{n=1}^{\infty} 1 = \infty$$

(b) Paying any finite sum, no matter how large, would leave you with an infinite expected profit. But most people would not be willing to pay more than a very modest sum.

(c) With $U(W) = W^{1/2}$, expected utility is

$$EU = \sum_{n=1}^{\infty} 2^{-n}\,(2^{n})^{1/2} = \sum_{n=1}^{\infty} (1/\sqrt{2})^{n} = \frac{1/\sqrt{2}}{1 - 1/\sqrt{2}} = \frac{1}{\sqrt{2} - 1} = 2.414$$

The sure sum of winnings that would give you the same utility is 5.828 .

(d) With $U(W) = \ln(W)$, we have $U(2^{n}) = n\ \ln(2) = 0.693\ n$. So expected utility is

$$EU = 0.693 \sum_{n=1}^{\infty} n\, 2^{-n} = 0.693 \times 2 = 1.386$$

This follows from

$$\sum_{n=1}^{\infty} n\, a^{n} = a \sum_{n=1}^{\infty} n\, a^{n-1} = a\, \frac{d}{da}\left[\sum_{n=1}^{\infty} a^{n}\right] = a\, \frac{d}{da}\left[\frac{a}{1-a}\right] = a\, \frac{1}{(1-a)^{2}}$$

and here $a = \frac{1}{2}$. The sure sum of winnings that would give you the same utility is $e^{1.386} = 4$.

(e) With $U(W) = -\ 1/W$, expected utility is

$$EU = -\ \sum_{n=1}^{\infty} 2^{-n}\,(2^{n})^{-1} = -\ \sum_{n=1}^{\infty} 4^{-n} = -\ \frac{1/4}{1 - 1/4} = -\ 1/3$$

The sure sum of winnings that would give you the same utility is 3.

(f) The coefficient of relative risk aversion is defined as $-\ W\, U''(W)/U'(W)$. This is just the elasticity of $U'(W)$ with respect to W (in numerical value). In these examples, we have

$$U'(W) = \tfrac{1}{2}\, W^{-1/2}, \qquad W^{-1}, \quad \text{and} \quad W^{-2}$$

respectively. Therefore the corresponding coefficients of RRA are 1/2, 1 and 2.

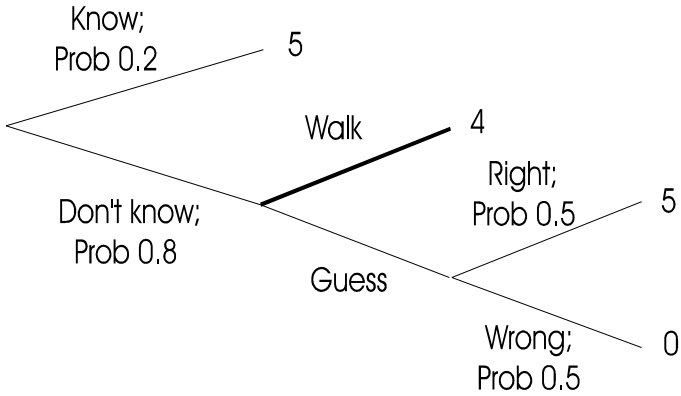
Answer 2:

(a) The successive utility levels are

$$U(32000) = \log_2(1) = 0, \quad U(64000) = \log_2(2) = 1, \quad U(128000) = \log_2(4) = 2.$$
$$U(256000) = \log_2(8) = 3, \quad U(512000) = \log_2(16) = 4, \quad U(1024000) = \log_2(32) = 5$$

(b) Label the levels 0 (the starting or \$32,000 level) to 4 (the \$512,000 level).

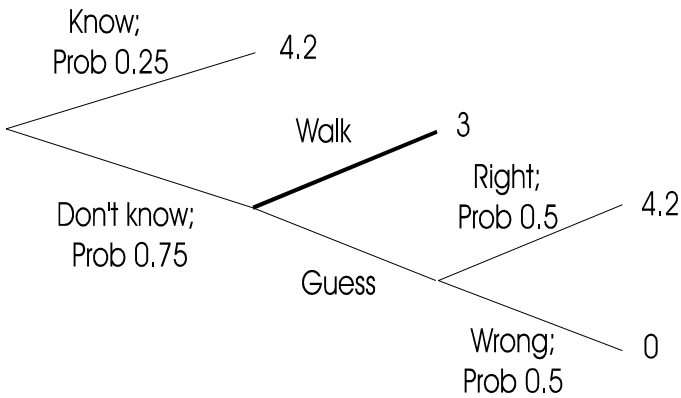
When you have reached level 4, you face the following tree:



The payoffs at the various possible end-points are utilities.

We solve this backward. Your expected utility from guessing is $0.5 \times 5 + 0.5 \times 0 = 2.5$. This is less than the 4 you get if you walk. So walking is optimal (shown by thickening that branch). Therefore, on reaching level 4 and getting the next question, with probability 0.2 you will know the answer, give it, and get 5, but with probability 0.8 you will not know the answer, will walk (that is optimal) and get 4, for an expected utility of $0.2 \times 5 + 0.8 \times 4 = 4.2$.

When you have reached level 3, you face the following tree:



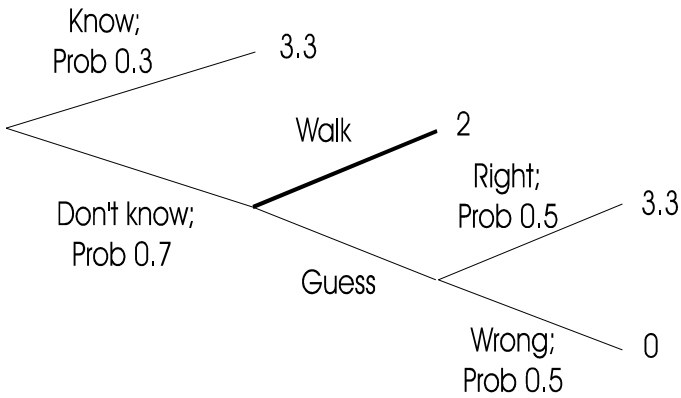
Here the payoffs 4.2 arise when you answer correctly (either because you know the answer or when you make a lucky guess) and proceed to level 4. We use the calculation of level-4 decisions and payoffs just made and substitute it into this – that is the backward induction logic.

The crucial idea is that the value of reaching level 4 is not just the utility of the prize amount at that level, namely $U(512,000) = 4$. One must include the possibility that one might win an even bigger prize. To find the correct value, therefore, one must solve out the later stages of the game. That is why one must proceed backward.

Note that the 4.2 must be substituted in two places – one where you know the answer, and the second where you make a lucky guess – because both of them take you to level 4 from where your expected utility is 4.2 . The second is important in making your walk/guess decision.

Your expected utility from guessing is $0.5 \times 4.2 + 0.5 \times 0 = 2.1$. This is less than the 3 you get if you walk. So walking is optimal (shown by thickening that branch). Therefore, on reaching level 3 and getting the next question, with probability 0.25 you will know the answer, give it, and get 4.2, but with probability 0.75 you will not know the answer, will walk (that is optimal) and get 3, for an expected utility of $0.25 \times 4.2 + 0.75 \times 3 = 3.3$.

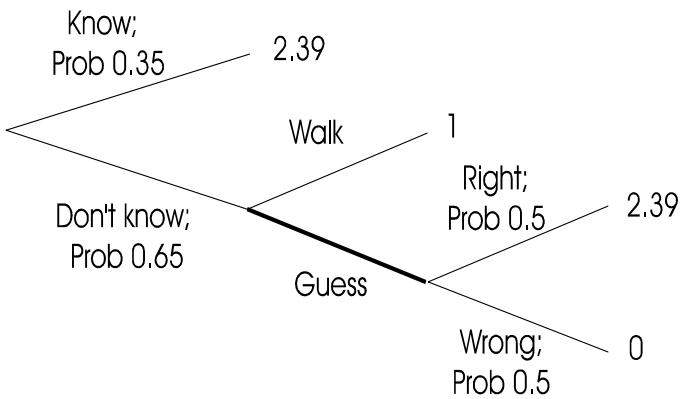
When you have reached level 2, you face the following tree:



Here the payoffs 3.3 are the result of now-familiar backward induction logic.

Your expected utility from guessing is $0.5 \times 3.3 + 0.5 \times 0 = 1.65$. This is less than the 2 you get if you walk. So walking is optimal (shown by thickening that branch). Therefore, on reaching level 2 and getting the next question, with probability 0.3 you will know the answer, give it, and get 3.3, but with probability 0.7 you will not know the answer, will walk (that is optimal) and get 2, for an expected utility of $0.3 \times 3.3 + 0.7 \times 2 = 2.39$.

When you have reached level 1, you face the following tree:

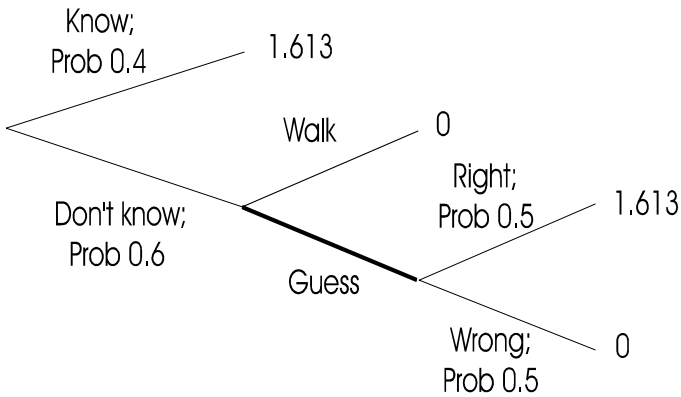


Here the payoffs 2.39 follow from backward induction once again.

Your expected utility from guessing is $0.5 \times 2.39 + 0.5 \times 0 = 1.195$. This is now more than the 1 you get if you walk. So guessing has become optimal (shown by thickening that branch). Therefore, on reaching level 1 and getting the next question, with probability 0.35 you will know the answer, give it, and get 2.39, but with probability 0.65 you will not know the answer, will guess (that is optimal) and get 1.195, for an expected utility of $0.35 \times 2.39 + 0.65 \times 1.195 = 1.613$.

Observe how "guess" has now become optimal instead of "walk". You might have thought that a correct guess would take you to the \$128,000 level, and $U(128,000) = 2$, so the expected utility from guessing should be $0.5 \times 2 + 0.5 \times 0 = 1$, which would keep you indifferent between walking and guessing. But the true expected utility of reaching the \$128,000 level is $2.39 > 2$. The excess 0.39 captures the expected value of going on to get even bigger prizes. This is where backward induction is crucial.

Finally, at the initial level 0, you face the following tree:



Here the payoffs 1.613 are the result of the usual backward induction logic.

Your expected utility from guessing is $0.5 \times 1.613 + 0.5 \times 0 = 0.8066$. This is now more than the 0 you get if you walk. So guessing is optimal (shown by thickening that branch). Therefore, on getting your first question, with probability 0.4 you will know the answer, give it, and get 1.613, but with probability 0.6 you will not know the answer, will guess (that is optimal) and get 0.8066, for an expected utility of $0.4 \times 1.613 + 0.6 \times 0.8066 = 1.129$.

Note that at this point, guessing would be optimal even if your chances of guessing correctly were less than 50 percent – any weighted average of 1.613 and 0 is bigger than 0. That is why the guess at the \$32,000 level is a "free guess", as the contestants recognize (or the host points out) in the show.

Thus your optimal strategy is to guess if you don't know the answers to the \$64,000 and \$128,000 questions, but to walk if you don't know the answers to the \$256,000, \$512,000 or \$1,024,000 questions. (I have simplifed the game drastically by having only 2 possible answers instead of 4 at each level, and leaving out the "lifelines" feature. Also log vN-M utility has a very specific level of risk aversion. Nonetheless, the guess-walk split found here corresponds to the one I have seen used by most people on the actual show once they have reached the \$32,000 level.)

(c) Your expected utility at the initial level is 1.129. This corresponds to winning $32,000 \times 2^{1.129} \approx 70,000$.