ECO 305 – Fall 2003
Precepts Week 11

Question

In this question, you are an investor with an initial wealth of 1 (in units of millions of dollars). You can allocate non-negative fractions of it among different assets as indicated in the various parts below (no leveraging or selling short).

(a) First suppose that you have available just two asset, namely shares of stock in firms called Nova and Supernova. Your analysis tells you that the expected return on Nova stock will be 100 per cent, so each dollar invested in Nova will get you back an expected value of 2 dollars. But there is risk, and the standard deviation is 1 per dollar invested. The expected return on Supernova is 2 (so each dollar invested gets you back 3), with a standard deviation of 1. The returns to the two assets have zero correlation with each other.

Write down the expressions for the expected return μ and the variance σ^2 of a portfolio that has a fraction z of your wealth in Supernova and the rest, $(1 - z)$, in Nova (no cash). What mixture between Nova and Supernova stocks minimizes the variance of the portfolio? What is the minimum variance?

(b) Next suppose you can also hold some of your wealth in cash, which has zero return and zero variance – you simply preserve your cash dollars for sure. Writing μ for the expected return on your portfolio and σ for its standard deviation, your objective is to maximize

$$\mu - \tfrac{1}{2}\, a\, \sigma^2\,,$$

where a is a measure of your risk aversion. Show that your optimal policy is as follows:

If $a < 1$, put all of your wealth in Supernova. If $1 < a < 3$, choose an appropriate mix of Nova and Supernova stocks, but no cash (the optimal mix varies with a). If $a > 3$, mix between cash and a risky portfolio that combines Nova and Supernova shares in the precise proportion of 1:2 that is independent of a; the optimal mix between cash and this combination depends on a.

(c) Why do you ever hold positive quantities of Nova stock, even though Supernova "dominates" it, yielding a higher expected return with the same variance?

(d) Illustrate your solution in the usual figure with the expected return of an asset or combination of assets on the vertical axis, and the standard deviation on the horizontal axis.

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Solution

(a) If you invest z (million dollars) in Supernova and $(1 - z)$ in Nova, the expected return will be

$$\mu = 2\,z + (1 - z) = 1 + z \tag{1}$$

and the variance

$$\sigma^2 = z^2 + (1 - z)^2 = 1 - 2\ z + 2\ z^2 \tag{2}$$

The FONC for minimization of σ^2 is

$$\frac{d\sigma^2}{dz} = -2 + 4\,z = 0$$

and the SOSC is obviously satisfied, so variance is minimized when $z = 1/2$. The minimum variance is $1/2$.

Alternatively, we can use (1) in (2)to express the relationship between mean and variance:

$$\sigma^2 = 1 - 2\ (\mu - 1) + 2\ (\mu - 1)^2 = 2\ \mu^2 - 6\ \mu + 5 \tag{3}$$

Then choose μ to minimize σ^2. The FONC is

$$\frac{d(\sigma^2)}{d\mu} = 4\ \mu - 6 = 0$$

and the SOSC is obviously satisfied. So the variance is minimized when $\mu = 3/2$, which corresponds to $z = 1/2$.

(b) When cash is available, suppose you invest a fraction z_n of your wealth in Nova and a fraction z_s in Supernova, leaving $(1 - z_n - z_s)$ in cash, the expected return on your portfolio is $z_n + 2\,z_s$ and its variance is $(z_n)^2 + (z_s)^2$. So you want to maximize

$$z_n + 2\ z_s - \tfrac{1}{2}\,a\,[\,(z_n)^2 + (z_s)^2\,]$$

subject to the constraints (because no short selling or leveraging is permitted)

$$z_n \geq 0, \qquad z_s \geq 0, \qquad z_n + z_s \leq 1$$

First suppose none of the constraints are binding. The FONCs are

$$1 - a\,z_n = 0, \qquad 2 - a\,z_s = 0$$

therefore

$$z_n = 1/a, \qquad z_s = 2/a, \qquad 1 - z_n - z_s = 1 - 3/a$$

The constraints are indeed non-binding if $a \geq 3$, so in this range of a this must be the optimum.

Note that $z_n/z_s = 1/2$. So regardless of a (in the range $a \geq 3$ of course), the optimal portfolio has a 1:2 mixture between Nova and Supernova. With a fraction $(a-3)/a$ going to cash, we have the rest, fraction $3/a$, going to this combination.

If $a < 3$, the constraint requiring you to hold non-negative cash is binding. So in this case let us re-solve the problem with the constraint $z_n + z_s \leq 1$ (while still ignoring the other two constraints $z_n \geq 0$ and $g_s \geq 0$). This produces the Lagrangean

$$z_n + 2\, z_s - \tfrac{1}{2}\, a\, [\, (z_n)^2 + (z_s)^2\,] + \lambda\, [1 - z_n - z_s]$$

leading to the FONCs

$$1 - a\, z_n - \lambda = 0, \qquad 2 - a\, z_s - \lambda = 0$$

Therefore

$$z_n = (1-\lambda)/a, \qquad z_s = (2-\lambda)/a$$

and

$$1 = z_n + z_s = (3 - 2\,\lambda)/a, \qquad \text{so} \qquad \lambda = (3-a)/2$$

Substituting,

$$z_n = (a-1)/(2\,a), \qquad z_s = (a+1)/(2\,a)$$

Observe that, to ensure $\lambda \geq 0$, we need $a \leq 3$, which is exactly the situation in which we were led to examine this case. But a legitimate solution must also have $z_n \geq 0$, for which we need $a \geq 1$.

If $a < 1$, the constraint $z_n \geq 0$ becomes binding. Then we must set $z_n = 0$ and get $z_s = 1$. (Of course one must check all the proper Kuhn-Tucker conditions, which I will now leave to you.)

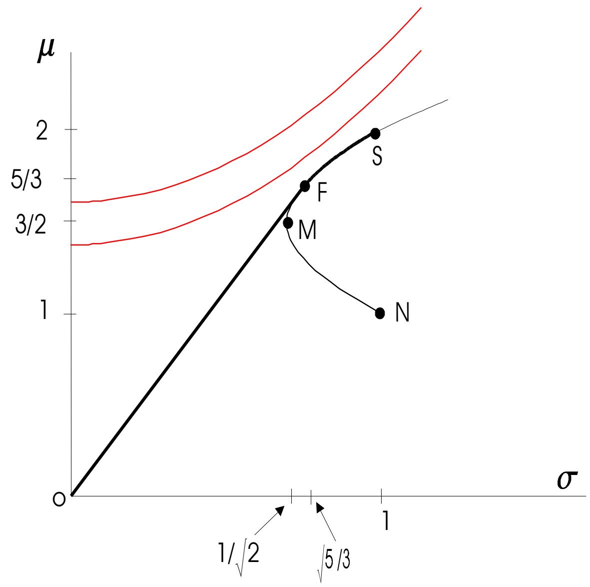
(c) You invest in the seemingly inferior Nova stock because it reduces the total risk by diversification.

(d) All this is illustrated in the figure at the top of the next page. The points O, N and S represent the (σ, μ) combinations for cash, Nova and Supernova respectively. The curve going from S to N represents the combinations for portfolios that mix just the two stocks in varying proportions as in part (a). The point M is the minimum-variance combination; note that for it the variance is $1/2$ so the standard deviation is $1/\sqrt{2}$. The continuation of the curve beyond S has relevance if you can sell Nova stock short and use the proceeds to by Supernova stock.

The point F represents the portfolio that has Nova and Supernova stocks in the proportions 1:2. Thus its expected rate of return is $(1/3) \times 1 + (2/3) \times 2 = 5/3$, with variance $(1/3)^2 \times 1 + (2/3)^2 \times 1 = 5/9$, or standard deviation $\sqrt{5}\,/3$.

The slope of the curve at any point is found by differentiating its equation (3):

$$2\,\sigma = (4\,\mu - 6)\,\frac{d\mu}{d\sigma}, \quad \text{or} \quad \frac{d\mu}{d\sigma} = \frac{\sigma}{2\,\mu - 3}$$



So the slope at the point F is

$$\frac{\sqrt{5}\,/3}{2\times 5/3-3}=\sqrt{5}$$

and since F $=(\sqrt{5}\,/3, 5/3)$, the slope of the line OF is

$$\frac{5/3}{\sqrt{5}\,/3}=\sqrt{5}$$

so the line OF is tangential to the curve.

Then the line OF and the segment from F to S of the curve comprise the efficient frontier of portfolio choices when short sales or leveraged purchases are not permitted.

The contours of your objective function (indifference curves) are given by

$$\mu-\tfrac{1}{2}\,a\,\sigma^2=k, \quad \text{or} \quad \mu=k+\tfrac{1}{2}\,a\,\sigma^2$$

These have the parabolic form shown. The slope of the indifference curve passing through the general point (σ,μ) is

$$d\mu/d\sigma=a\,\sigma$$

At the point F $=(\sqrt{5}\,/3, 5/3)$, this equals $a\sqrt{5}\,/3$. If that is greater than the slope of OF, namely $\sqrt{5}$, then the indifference curve through F will be steeper than OF, so the optimum will be somewhere along the segment OF, at a point of tangency with an indifference curve. Thus when $a>3$ the optimum portfolio is a mixture of cash and the special 2:1 stock portfolio.

At the point S $=(1,2)$, the slope of the frontier curve FS is $1/(2\times 2-3)=1$, and that of the indifference curve passing through S equals a. Therefore if $a<1$, the indifference curve is flatter and we get an end-point optimum at S.

Finally, if $1<a<3$, at F the indifference curve is flatter than the efficiency frontier, but at S the indifference curve is steeper than the efficiency frontier, so the optimum is at a point of tangency along the curve FS, where no cash is held.