ECO 305 – Fall 2003
Precepts Week 12 – Questions

This week's topic is incentive payments. We will consider in general terms an example where the issue arises, and then some elementary theory.

Question 1:

I recently had to correct the page proofs of a 600-page book. (For those of you who don't know what this entails, it means reading in parallel my manuscript and the typesetter's rendering of it, to correct the errors that the typesetter had committed.) I was very busy with preparation of lecture, precept, and problem set materials for ECO 305; therefore I had to hire a student to do the proofreading for me.

When it came to devising a contract of work and payment for this job, I had to recognize some problems of information asymmetry. What are they? How can a payment scheme attempt to cope with them?

Question 2:

Suppose you are the owner of a firm, and one of your managers is in charge of a project. If the manager makes effort x, your output will be

$$y = x + \epsilon \tag{1}$$

where ϵ is distributed normally with mean zero and variance v. You cannot observe effort directly, but can observe output. More importantly, output can be verified by a third party such as a court that might be called upon to enforce a contract. Therefore you can credibly promise the manager a payment $z(y)$ that is a function of the output y.

The manager gets some disutility from effort, and is liable to shirk given a chance, blaming any bad results on bad luck. But, since output is partially a reflection of effort, you can make some component of the manager's compensation depend on the output, and thereby induce him or her to make more effort. We want to find the optimal scheme of this kind from your perspective. We consider a special example (with a mean-variance objective function and a linear payment scheme), which conveys out some basic intuition.

Suppose therefore that the function $z(y)$ takes the form

$$z(y) = k + m\, y \tag{2}$$

Thus k is a base salary and m is the bonus or incentive payment per unit of output. We want to find the optimal values of k and m.

Suppose that you want to maximize the expected value of your ouput minus what you pay the manager, minus a correction for risk:

$$\mathrm{E}[y - z] - \tfrac{1}{2}\, A\ \mathrm{V}[y - z] = \mathrm{E}[(1 - m)\, y] - k - \tfrac{1}{2}\, A\ \mathrm{V}[(1 - m) y] \tag{3}$$

where A is your coefficient of risk-aversion.

The manager is risk-averse, and has the utility function

$$\mathrm{E}[z] - \tfrac{1}{2}\, a\, \mathrm{V}[z] - \tfrac{1}{2}\, c\, x^2 \tag{4}$$

where a is his coefficient of risk-aversion, and the last term is the disutility of effort.

(a) Substitute from (1) into (2) and calculate the mean and the variance of z as a function of k and m.

(b) Substitute the result into (4) to express the manager's utility as a function of x, the incentive parameters k, m, and the behavioral parameters a, c.

(c) The manager chooses x to maximize this. Solve for his optimal choice and substitute back to find an expression of the manager's resulting ("indirect") utility as a function of the incentive parameters k, m (and the behavioral parameters a, c).

(d) The manager can find other work that gets him utility u_0. Therefore to attract him to work for you, you must offer him at least this much utility. Express this as a constraint on your choice of k, m.

This is called the "participation constraint". (The other constraint on your choice, called the "incentive compatibility constraint," namely that you cannot control x directly but must let the manager choose it to maximize his own utility, has already been incorporated by the calculation in (c).)

(e) Use your work in parts (a) and (c) to express your own objective (3) as a function of k, m (and of course the behavioral parameters).

(f) Choose k and m to maximize the expression you found in (e), subject to the participation constraint in (d), to find your optimal choices of k and m. Specifically, show that

$$m = \frac{1 + A\,c\,v}{1 + (a + A)\,c\,v} \tag{5}$$

(g) Interpret this result in economic terms.

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Precepts Week 12 – Solutions

Question 1:

I want the student to catch all the typesetting errors there may be, but I cannot be sure of that unless I do the checking myself, which defeats the whole purpose of hiring the student. The student's effort is unobservable - she takes away the materials and comes back a week later to tell me of the errors she has found. What is worse, even the outcome cannot be observed immediately. I will find out about any error she failed to catch only when some other reader tells me about them. So the student has the temptation to shirk – just hold on to the materials for a few days and then tell me that there are no errors. So I cannot offer a her a fixed flat sum for the job.

But if I offer a piece rate (so much per error she finds), she may worry that the typesetter has done a perfect job, so she may have to spend a week or more on the work and get no money at the end of it. She will be reluctant to take the job on these terms.

The compensation scheme has to be a compromise between the two extremes – a flat sum plus a bonus per error she discovers. This should give her enough assurance of the total compensation to make the job attractive enough, and give her enough incentive to attempt a thorough reading.

My solution was a dollar per page ($600 total) as a flat sum, plus $1 per error found (there were 274). I don't claim it was fully optimal or the best deal I could have got. I am waiting to see if there are any remaining errors. But the general principle sets the stage for the theoretical work of the next question.

Question 2:

(a) $z = k + m\,x + m\,\epsilon$. Only ϵ is random, and has zero mean and variance v. Therefore

$$\mathrm{E}[z] = k + m\,x, \qquad \mathrm{V}[z] = m^2\ v$$

(b) The manager's objective function becomes

$$k + m\,x - \tfrac{1}{2}\,a\,m^2\ v - \tfrac{1}{2}\,c\ x^2$$

(c) The FONC for x is

$$m - c\,x = 0, \qquad \text{implying } x = m/c$$

The SOSC is satisfied. This conforms to intuition: the manager makes more effort if the marginal bonus payment is higher, and less if the disutility parameter is higher. Note that the manager's choice of x does not depend on k; that is only an additive constant in his objective.

Substituting back, the manager's indirect utility function is

$$k + \tfrac{1}{2}\,m^2/c - \tfrac{1}{2}\,a\,v\,m^2$$

(d) The participation constraint is

$$k + \tfrac{1}{2}\,m^2/c - \tfrac{1}{2}\,a\,v\,m^2 \geq u_0$$

(e) In your objective function,

$$\mathrm{E}[(1-m)y] = (1-m)\ x = (1-m)\,m/c$$

and

$$\mathrm{V}[(1-m)y] = (1-m)^2\ v$$

You want to maximize

$$(1-m)\,m/c - k - \tfrac{1}{2}\,A\ (1-m)^2\ v$$

subject to the participation constraint. You can substitute out for k and express the objective as a function of m (and other parameters that are not your choice variables). But let us do Lagrange for a change. The Lagrangian is

$$(1-m)\,m/c - k - \tfrac{1}{2}\,A\,(1-m)^2\;v + \lambda\,[\,k + \tfrac{1}{2}\,m^2/c - \tfrac{1}{2}\,a\,v\,m^2 - u_0\,]$$

The FONCs with respect to k and m are respectively

$$-1 + \lambda = 0$$

and

$$(1-2\,m)/c + A\,(1-m)\;v + \lambda\,[\,m/c - a\,v\,m\,] = 0$$

Using $\lambda = 1$ in the second, it simplifies to

$$(1-m)/c + A\,v - (A+a)\,v\,m = 0$$

or

$$m = \frac{1 + A\,c\,v}{1 + (A+a)\,c\,v}$$

as required.

(g) Interpretations:

[1] If $a = 0$, then $m = 1$. The manager's bonus will be equal to the owner's valuation of each unit of output. Therefore a risk-neutral manager will be given 100% incentive to exert effort. Of course the k will adjust and will typically be negative – the owner will in effect sell the whole operation to the manager, who as owner-manager will exert optimal effort.

[2] In general, $m < 1$. Some power of incentive is sacrificed in exchange for some risk-sharing. This is a "second-best". The effort is less than in the ideal first-best, but the owner finds this optimal because the manager works for a sufficiently lower total expected compensation in return for shedding some risk.

To see this more explicitly, solve for k from the participation constraint:

$$k = u_0 - \tfrac{1}{2}\,m^2/c + \tfrac{1}{2}\,a\,v\,m^2$$

The higher is the manager's risk-aversion, the higher the base salary he must be offered in order to induce him to take the risk of working at this project.

[3] This effect is more important (the power of incentives and the effort in the second-best are less) when the error variance v is larger, that is, when the output is a less accurate measure of the underlying effort.

[4] The effect is more important the larger is c the effort cost parameter.

More general models of this kind: [1] examine when linear schemes are truly optimal, and when they are not, how nonlinear schemes (for example step-functions) can do better, [2] allow several dimensions of effort and output and see how the incentive schemes for different types of output interact, [3] consider how repeated interactions and comparisons across several agents allow sharper incentives.

Perhaps most importantly in practice, the owner's outcome y is itself not verifiable and cannot serve as the basis for a contract specifying payment. In the proofreading problem, my true objective is to have an error-free book. But that cannot be known for certain for many years, by which time the student may be untraceable. So I have to condition payment on an imperfect proxy for my true objective. For a very good theoretical analysis of this situation, read two papers by George Baker (HBS professor):

Incentive contracts and performance measurement. *Journal of Political Economy* vol. 100, June 1992, pp. 598–614.

Distortion and risk in optimal incentive contracts. *Journal of Human Resources* vol 37, Fall 2002, pp. 728–751.