ECO 305 - Fall 2003
Microeconomic Theory - A Mathematical Approach
Problem Set 1 - Answer Key

The distribution was as follows:

Range	Number
100	1
90–99	48
80–89	23
70–79	10
60–69	6
< 60	2

This is a very encouraging start; try to keep up this standard of work and even improve on it. Please read the answer key carefully. Even if you scored very high, the key will usually mention a couple of interpretations and additional twists that have educational value, and more important, may be used on exams. Where you lost points, your answers will have some brief annotation of the reason, which you can understand in more detail from the answer key. If a question still remains, ask Arnaud Costinot, who graded this problem set.

Remember that grading is on the full 0–100 scale, not an everyone-gets-smiley-faces 70–100 scale. So even a grade below 60 does not necessarily mean you are failing the course (in fact historically, failure has been very rare). But if you got a grade below 70, it does suggest that you could do better. Make sure to get your difficulties resolved early. You can use anyone's office hours, not just your own preceptor's. Conversely, if you did very well, don't get complacent: there may be nowhere for you to go but down :-(.

COMMON ERROR – Several points were lost through sheer carelessness, including forgetting to calculate some simple answer that was required e.g. the resulting GPA in Question 3. Please avoid such simple mistakes in the future. Read the questions completely and double-check to make sure you have answered them all.

Question 1: (total 45 points)

COMMON ERRORS – Lack of explanations. Remember that the question told you to do the calculus to solve for the critical points etc., and not merely eyeball them from the graph. More generally, in math problems it is important to show the steps of your work; see the general Instructions for Problem Sets, para 5.

(a) (5 points) Figure 1 shows the graph of the function, here drawn using Mathematica and more accurately than you need to show. The points suggested for further investigation are marked with bullets.

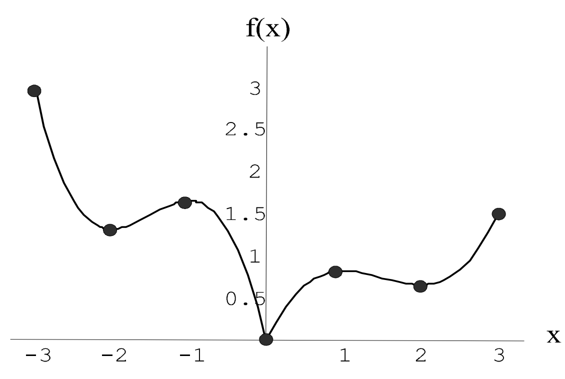

Figure 1: Question 1 (a)

(b) The function is defined by different formulas in different intervals so we need to check each separately.

(5 points) In (-3,0), $f'(x) = -4 - 6\,x - 2\,x^2 = -2\,(x^2 + 3\,x + 2) = -2\,(x+1)\,(x+2)$. So $f'(x) = 0$ has roots $x = -1$ and $x = -2$.

(5 points) $f''(x) = -6 - 4x$. Then $f''(-1) = -2 < 0$ so there is a local maximum at $x = -1$, and $f''(-2) = 2 > 0$ so there is a local minimum at $x = -2$. The values of the function at these points are $f(-1) = 4 - 3 + \frac{2}{3} = \frac{5}{3} = 1.667$ and $f(-2) = 8 - 12 + \frac{16}{3} = \frac{4}{3} = 1.333$. (Calculating values is important for subsequent check of global maxima and minima.)

(5 points) In (0,3), $f'(x) = 2 - 3\,x + x^2 = (x-1)\,(x-2)$, so $f'(x) = 0$ has roots $x = 1$ and $x = 2$.

(5 points) $f''(x) = -3 + 2\,x$. Then $f''(1) = -1 < 0$ so there is a local max at $x = 1$, and $f''(2) = 1 > 0$ so there is a local min at $x = 2$. The values of the function at these points are $f(1) = 2 - \frac{3}{2} + \frac{1}{3} = \frac{5}{6} = 0.833$, and $f(2) = 4 - 6 + \frac{8}{3} = \frac{2}{3} = 0.667$.

(c)(5 points) $f'(0) = -4$ from the left and 2 from the right. Therefore the function is not differentiable at 0. The signs of the derivatives meet the conditions for a local minimum at $x = 0$. The value is $f(0) = 0$.

(d) (5 points) $f'(-3) = -4 + 18 - 18 = -4 < 0$. This fulfills the condition (necessary nd sufficient) for a local max at a left endpoint. The value is $f(-3) = 12 - 27 + 18 = 3.000$.

(5 points) $f'(3) = 2 - 9 + 9 = 2 > 0$. This fulfills the condition (necessary and sufficient) for a local max at a right endpoint. The value is $f(3) = 6 - \frac{27}{2} + 9 = 1.500$.

(e) (5 points) Comparing values directly, the global max is at $x = 3$, with $f(3) = 3$, and the global min is at $x = 0$, with $f(0) = 0$.

Question 2: (total 35 points)

COMMON ERRORS – A few people did not draw a graph, and their subsequent reasoning especially for second-order conditions was faulty or missing as a result. A few people drew a 3-D graph, which is fine if it is clear. Some who drew a 2-D graph did not show the direction in which the objective function is increasing; this is important when sorting out a max from a min. Many had an uncertain grasp of the concept of relative convexity of indifference curves and objective function contours. We have been very generous on this point (no points were deducted if a clear graph was presented) because it is new, but you should gradually understand and express it better.

(a) (5 points) Figure 2 shows the sketch. It shows that the tangency gives a maximum, so Lagrange's method can be used.

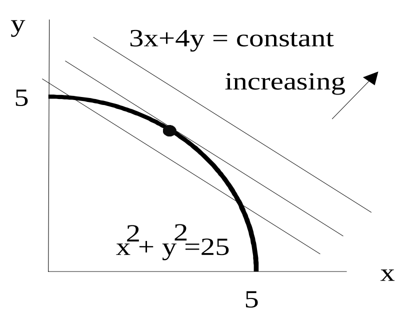

Figure 2: Question 2 (a)

(10 points) The Lagrangian is

$$L = 3\,x + 4\,y + \lambda\,(25 - x^2 - y^2)\,.$$

The first-order conditions are

$$3 - 2\,\lambda\,x = 0 = 4 - 2\,\lambda\,y\,.$$

Therefore $4\,x = 3\,y$. Substituting in the constraint,

$$25 = (3\,y/4)^2 + y^2 = 25\,y/16\,.$$

Therefore $y = 4$ and $x = 3$. The maximized value of the objective function is $3 \times 3 + 4 \times 4 = 25$.

(b) (5 points) Figure 3 shows the sketch. It shows that the tangency gives a minimum, so Lagrange's method can be used.

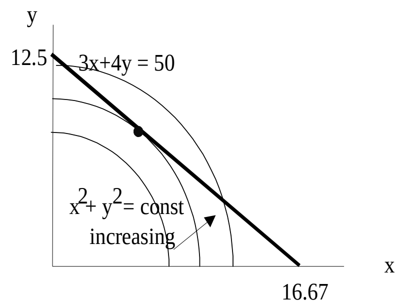

Figure 3: Question 2 (b) and (c)

(10 points) The Lagrangian is (note the way the constraint is incorporated to get a positive Lagrange multiplier):

$$L = x^2 + y^2 + \mu\,(50 - 3\,x - 4\,y)\,.$$

The first-order conditions are

$$2\,x - 3\,\mu = 0 = 2\,y - 4\,\mu\,.$$

Therefore $4\,x = 3\,y$. Substituting in the constraint,

$$50 = 3\,(3\,y/4) + 4\,y = 25\,y/4\,.$$

Therefore $y = 8$ and $x = 6$. The minimized value of the objective function is $6^2 + 8^2 = 100$.

(c) (3 points) Figure 3 serves once again. But now, since we want to maximize the function whereas the tangency gives a minimum, Lagrange's method is not applicable.

(2 points) We must simply compare the values at the two end-points, and the maximum is $(16.67)^2 = 277.78$.

Question 3: (total 20 points)

COMMON ERRORS – The vast majority did not check or even mention the second-order conditions, perhaps because the value of the GPA came out as 4.0 and so it had to be max. Still a mathematical check is necessary. No points were taken away this time, but may be in the future.

(10 points for finding the optimum) The function is strictly concave (obvious by visualizing the graph of the square root function). The constraint function $15 - F - M (\geq 0)$ is linear. So the curvature conditions are met. The marginal effect at zero of each activity on G is infinite so there cannot be a boundary solution with one of the variables equal to zero. Therefore we will have a unique regular maximum characterized by the Lagrange FONCs. At this point we do not expect you to get this detail fully correct, but up to 2 points were deducted for overlooking the matter entirely or getting it hopelessly wrong. Gradually you should learn to figure these things out and state them fully correctly.

The Lagrangian is

$$L = \frac{4}{5\sqrt{3}}\left[\sqrt{F} + 2\sqrt{M}\right] + \lambda\,[15 - F - M]$$

The FONCs are

$$\frac{4}{5\sqrt{3}}\,\tfrac{1}{2}\,F^{-1/2} - \lambda = 0, \qquad \frac{4}{5\sqrt{3}}\,2\,\tfrac{1}{2}\,M^{-1/2} - \lambda = 0.$$

Therefore

$$2\,M^{-1/2} = F^{-1/2}, \qquad \text{or} \qquad M = 4\,F.$$

Using the time-budget constraint, the optimum choices are $F = 3$ and $M = 12$, yielding the GPA

$$G = \frac{4}{5\sqrt{3}}\,[\sqrt{3} + 2\sqrt{12}\,] = \frac{4}{5\sqrt{3}}\,[\sqrt{3} + 4\sqrt{3}\,] = 4.0$$

(Yeah!). (6 points)

The shadow value of study time is

$$\lambda = \frac{4}{5\sqrt{3}}\,\tfrac{1}{2}\,3^{-1/2} = \frac{4}{30} = \frac{2}{15} \approx 0.133$$

(4 points)

Note: [1] Since Princeton does not allow GPA to exceed 4.0, this should be interpreted as the GPA you would lose per unit of study time below 15 hours. I hesitate to say "one hour less" because that is not an infinitesimal change and here we are talking about a derivative. Perhaps one should say that you would lose $0.133/60 \approx 0.00222$ GPA points per minute less of study. [2] This is the *marginal* product of a study hour. It is not equal to the *average* product of your study hours, which is $4/15$. In fact the marginal product is exactly half of the average product. Can you guess why? You will find out when we do production functions.