ECO 305 - Fall 2003
Microeconomic Theory - A Mathematical Approach
Problem Set 3 - Answer Key

This was graded by Arnaud Costinot, and the distribution was as follows:

Range	Number
100	9
90–99	44
80–89	16
70–79	10
60–69	2
< 60	4

A general "error" – please remember to write your precept time and simplify our task of sorting out the graded answers in the right folders.

Question 1: (total 40 points)

COMMON ERRORS: (1) Some of you solved a utility maximization problem instead of the expenditure-minimization problem that is needed. This is OK provided you then invert the indirect utility function to get the expenditure function, and some did not do this. (2) In (b)(2), several people said that $M = U$ if $P/R = 1$ (should be $M = P\,U = R\,U$). Some got the right optimum quantities but simply omitted to write down the final step – the expenditure function itself. (3) Very few of you got the neatly simplifed forms of the expenditure functions below. This is not an "error" as such; no points lost. But you missed the opportunity of seeing for yourself the neat "duality" between the two functional forms in each part. See the discussion after the solutions below.

(a)(1) (12 points) The hint tells you to write the constraint in the tranformed form

$$X^{-1} + Y^{-1} = u^{-1}$$

where u is the target utility level. The Lagrangian is

$$L = P\,X + R\,Y + \mu\,[\,u^{-1} - X^{-1} - Y^{-1}\,]$$

(Doesn't matter if you used the opposite signs in the second term; you will just end up working with $-\mu$ instead of μ.) The FONCs are

$$P = \mu\;X^{-2}, \qquad R = \mu\;Y^{-2}\,.$$

Therefore

$$X^{-1} = P^{1/2}\;\mu^{-1/2}, \qquad Y^{-1} = R^{1/2}\;\mu^{-1/2}\,.$$

Substitute into the constraint:

$$u^{-1} = X^{-1} + Y^{-1} = \mu^{-1/2}\;[\;P^{1/2} + R^{1/2}\;]\,,$$

or

$$\mu^{1/2} = u\;[\;P^{1/2} + R^{1/2}\;]\,.$$

Therefore

$$X = P^{-1/2}\;\mu^{1/2} = u\;[\;P^{1/2} + R^{1/2}\;]\;P^{-1/2}, \qquad Y = R^{-1/2}\;\mu^{1/2} = u\;[\;P^{1/2} + R^{1/2}\;]\;R^{-1/2},.$$

Finally, the minimized expenditure is

$$\begin{aligned} M^* &= P\,X + R\,Y = u\;[\;P^{1/2} + R^{1/2}\;]\;\{\,P^{1/2} + R^{1/2}\,\} \\ &= u\;[\;P^{1/2} + R^{1/2}\;]^2\,. \end{aligned}$$

(a)(2) (12 points) The hint tells you to write the constraint in the tranformed form

$$X^{1/2} + Y^{1/2} = u^{1/2}$$

where u is the target utility level. The Lagrangian is

$$L = P\,X + R\,Y + \mu\;[\,u^{1/2} - X^{1/2} - Y^{1/2}\,]$$

(Doesn't matter if you used the opposite signs in the second term; you will just end up working with $-\mu$ instead of μ.) The FONCs are

$$P = \tfrac{1}{2}\mu\;X^{-1/2}, \qquad R = \tfrac{1}{2}\mu\;Y^{-1/2}\,.$$

Therefore

$$X^{1/2} = \tfrac{1}{2}\mu\;P^{-1}, \qquad Y^{1/2} = \tfrac{1}{2}\mu\;R^{-1}\,.$$

Substitute into the constraint:

$$u^{1/2} = X^{1/2} + Y^{1/2} = \tfrac{1}{2}\mu\;[\;P^{-1} + R^{-1}\;]\,,$$

or

$$\tfrac{1}{2}\mu = u^{1/2}\;[\;P^{-1} + R^{-1}\;]^{-1}\,.$$

Therefore

$$X = P^{-2}\;[\tfrac{1}{2}\mu]^2 = u\;[\;P^{-1} + R^{-1}\;]^{-2}\;P^{-2}, \qquad Y = R^{-2}\;[\tfrac{1}{2}\mu]^2 = u\;[\;P^{-1} + R^{-1}\;]^{-2}\;R^{-2},.$$

Finally, the minimized expenditure is

$$\begin{aligned} M^*(P,R,u) &= P\,X + R\,Y = u\;[\;P^{-1} + R^{-1}\;]^{-2}\;\{\,P^{-1} + R^{-1}\,\} \\ &= u\;[\;P^{-1} + R^{-1}\;]^{-1}\,. \end{aligned}$$

General notes on part (a): Note how the expenditure function in one case has the same functional form as the utility function in other case – they are mutually "dual". Also, with years of trial and error, I have learned how to do this algebra in an easier or slicker way. Many of you probably used longer or more involved methods. That is fine; you can compare your solution to mine and gradually learn the slicker methods just as I did.

(b) The figure below shows, for each of the two cases, the indifference curve corresponding to the desired utility level, and a set of parallel budget line with slope numerically equal to the price ratio P/R.

(1) (8 points) The indifference curve has numerical slope 1. The budget line is shown steeper, corresponding to the case $P/R > 1$, that is, $P > R$. The expenditure-minimizing choice is $X = 0$, $Y = u$. (With perfect substitutes, you buy only the cheaper good.) So the expenditure is $u\,R$. If $P < R$, we would get $X = u$, $Y = 0$ and expenditure $= u\,P$. If $P = R$, any point on the line $X + Y = u$ is equally good, and expenditure $= u\,P = u\,R$. So the expenditure function is

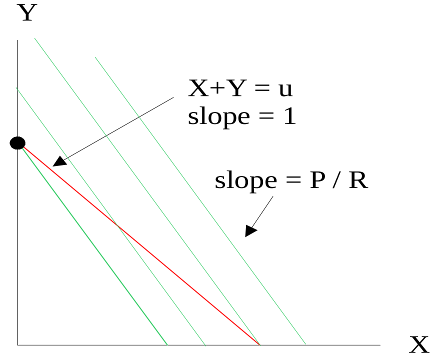
$$M^*(P,R,u) = u\;\min(P,R)\,.$$

(2) (8 points) The indifference curve is L-shaped, and the optimal choice is at its corner, with $X = Y = u$, regardless of the prices. Therefore

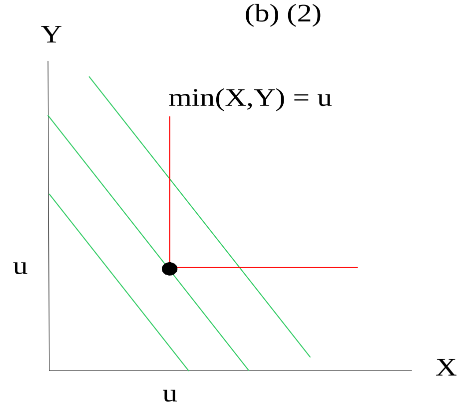
$$M^*(P,R,u) = u\,(P + R)\,.$$

General note on part (b): Observe how the two functional forms are again mutually "dual". One other point: the questions some of you raised during office hours or precepts often showed confusion about what is

(b) (1)



(b) (2)



wanted. When the indifference curves are straight lines, for example, some people asked how one could know the choice without knowing P and R. Of course the choice depends on P and R, and what is needed is a formula, or a function, showing how it depends. This is no different from part (a), where people successfully found such a formula. The only special thing about part (b) is that the formula has a kind of discontinuity; very different things happen depending on whether $P < R$ or $P > R$. But the principle is exactly the same.

A related issue: Many of you kept on drawing one budget constraint and several indifference curves. This is not a problem of utility maximization subject to a budget constraint, but the mirror-image problem of minimizing the cost of achieving a given utility level. So you should draw just one indifference curve, and that is the constraint. You should draw several parallel lines each representing a constant amount of expenditure; those are the level curves of your objective function namely the expenditure, or the "iso-expenditure contours". Of course you could solve the usual utility maximization problem subject to the budget constraint, find the indirect utility function, and then invert to get the expenditure function. If done correctly with the logic stated clearly, this gets full credit. (Likewise in part (a)).

Carrying on the idea of duality, if the utility function is Cobb-Douglas, so is the expenditure function and the prices have the same powers in the expenditure function as do the quantities in the utility function. The Cobb-Douglas function is "self-dual".

All these functions are special cases of the general family of "constant elasticity of substitution" functions.

Question 2: (total 30 points)

COMMON ERRORS: (1) Several of you did the Lagrange solutions in detail instead of just stating and using the Cobb-Douglas results. This is fine so long as you do it properly, including looking at the curvature of the indifference curves for second-order conditions, and explaining why there are no boundary solutions (utility is zero when one of the quantities is zero, but positive utility is achievable in the interior). (2) Some people approximated 10/3 by 3.5 and this led to wrong arithmetic later. (3) Some used a revealed preference argument for Fritz but did not give it correctly, simply saying that it was "not possible" or "not valid". The correct statement is that when all goods have positive marginal utility, an optimum cannot be in the interior of the budget set. (4) Some did not compare the two local optimal directly but simply took a guess (Pizza gets a higher power in Fritz's utility so he should eat more pizza. This is a valid intuition but needs to be confirmed by math; after all the power 4 might not be high enough. You could also get a clue from the last names of the two.) (5) Some good students also compared the two tangencies with the kink point. In this case that is pointless since a smooth indifference curve can never give a local optimum at an inward-pointing kink in the budget set. This is not an error, but something to learn.

Label the phone plan without the fixed fee Plan 1, and the other one Plan 2. Remember that phone usage is measured in units of "hundred minutes". Therefore the price of each unit under Plan 1 is 10 cents times $100 = \$10$, and the price under Plan 2 is $3\frac{1}{3}$ cents times $100 = \$\ \frac{10}{3}$. (Some people measured phone usage in minutes instead of hundreds of minutes. This is OK so long as you then transform the utility functions appropriately. It merely involves inserting a constant factor 100 in each utility function, and so has no effect on the outcome, but it must be shown for the work to be correct. One point off if you did not.

The budget constraint if Plan 1 is used is

$$10\ M + 10\ P = 100\,,$$

and the budget constraint if Plan 2 is used is

$$\tfrac{10}{3}\ M + 10\ P = 100 - 40 = 60\,.$$

Once a plan has been chosen, the budget constraint is a straight line and the utility function is Cobb-Douglas, and the standard theory of that situation applies. For each budget constraint there is a tangency solution. But one of the two will be a local and not a global maximum. So we find each local maximum in the usual way and then compare them directly. (10 points for correct formulation up to this point)

We know from standard Cobb-Douglas theory that if the utility function is

$$U(X,Y) = X^a\ Y^b$$

and the budget constraint is

$$P_x\ X + P_y\ Y = M\,,$$

then the optimum quantities are given by the constant budget share rule:

$$\frac{P_x\ X}{M} = \frac{a}{a+b}, \qquad \frac{P_y\ Y}{M} = \frac{b}{a+b}\,.$$

Consider Fritz first. If he chooses Plan 1,

$$\frac{10\ M_F}{100} = \frac{1}{1+4}, \ \frac{10\ P_F}{100} = \frac{4}{1+4}, \qquad M_F = 2,\ P_F = 8\,,$$

yielding utility $2 \times 8^4 = 8192$. If he chooses Plan 2,

$$\frac{(10/3)\ M_F}{60} = \frac{1}{1+4}, \ \frac{10\ P_F}{60} = \frac{4}{1+4}, \qquad M_F = 3.6,\ P_F = 4.8\,,$$

yielding utility $3.6 \times 4.8^4 = 1911$. So Fritz will choose Plan 1, talk for 200 minutes on the phone, and eat 8 pizzas. (10 points)

Next Gaby. If she chooses Plan 1,

$$\frac{10\ M_G}{100} = \frac{1}{1+1}, \ \frac{10\ P_G}{100} = \frac{1}{1+1}, \qquad M_G = 5,\ P_F = 5\,,$$

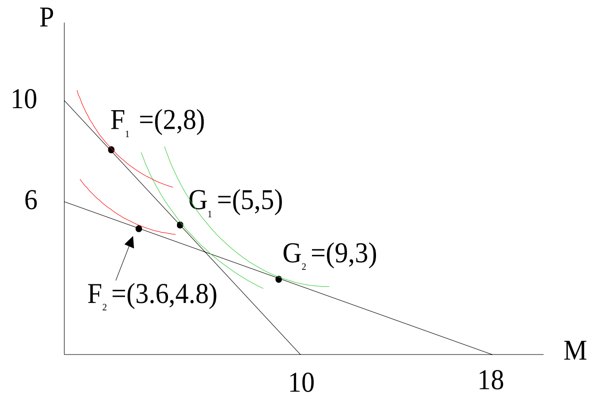
yielding utility $5 \times 5 = 25$. If she chooses Plan 2,

$$\frac{(10/3)\ M_G}{60} = \frac{1}{1+1}, \ \frac{10\ P_G}{60} = \frac{1}{1+1}, \qquad M_G = 9,\ P_F = 3\,,$$

yielding utility $9 \times 3 = 27$. So Gaby will choose Plan 2, talk for 900 minutes on the phone, and eat 3 pizzas. (10 points)

Additional information you were not asked to supply but would have found useful in your working: The figure shows the two budget constraints, and the two tangencies F_1 and F_2 for Fritz, G_1 and G_2 for Gaby. The indifference curves are not quantitatively accurate but only sketches. Note that each student has his or her own indifference map representing his or her own preferences. Curves on one person's map can intersect those on the other's without creating any contradiction. In fact at any point, Fritz's indifference curve through that point is flatter than Gaby's, because Fritz values talk relatively less than does Gaby.

Fritz's tangency on the Plan 2 budget constraint lies in the interior of the budget constraint for Plan 1. So there are several choices under Plan 1 that would give him more of both goods, therefore it is obviously



not going to be optimal for him to choose Plan 2. This is a valid argument, basically a "revealed preference" argument, leaving no need to compare utilities at the two tangencies. If you gave this argument correctly, you got full credit. Gaby's tangency for Plan 1 is not "obviously inferior" in this way, so a direct comparison is essential.

Note also that Fritz's utility numbers are much bigger than Gaby's, but this has no significance in this context since we are not comparing their well-being in any way. If for example we were considering the distribution of income between them, then we would pay attention to the way their utilities were comparably scaled, and re-scale them as appropriate, to reflect our judgment of their relative merits or needs.

Question 3: (total 30 points)

COMMON ERRORS: (1) Inconsistent conventions leading to wrong utility functions or budget constraints when there is a black market. See below. (2) Some said that at $G = 20$ and $R = 980$ the "MRS different from price so it cannot be an optimum" without recognizing the kink in the budget constraint when there is rationing. (See the figure below.) (3) Some write the MRS as a negative number. This is fine provided you then consistently compare it with the negative of the price ratio. I have generally used the convention that the MRS is to be interpreted as a numerical value, and then compared to the price ratio, also a positive number. You can use the other convention but then must do so consistently.

(a) (8 points) If she buys $G = 20$ at price 1, she has $R = 980$ to spend on other goods.

The expression for her MRS at a general point (G, R) is

$$\frac{dR}{dG} = \frac{\partial U/\partial G}{\partial U/\partial R} = \frac{0.05/G}{0.95/R} = \frac{R}{19\ G}\,.$$

At (20,980), this equals $980/380 = 2.58$. This is greater than 1, the price.

(b) (10 points) In the figure below, the budget constraint plus the ration constraint keep Della's choice constrained to the bent line ABC (the vertical line BC represents the ration constraint). Then it is obvious that B is optimal for her.

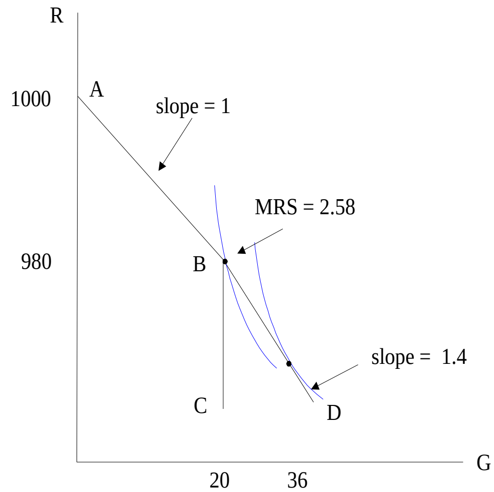
An intuitive way to start thinking about this is to note that Della's utility function is Cobb-Douglas. So in the absence of a rationing constraint, her expenditure proportion on gasoline would be 0.05/(0.05+0.95) = 0.05. Then she would buy $G = 0.05 \times 1000/1 = 50$ gallons. The ration is less than this, so the ration constraint is going to bind. Of course one needs to check the optimality more formally.

(c) (12 points) Now the budget constraint becomes the bent line ABD, where the slope of BD is 1.4, the black market price. Since the MRS at B exceeds the black market price, the optimum is going to be at a point of tangency of an indifference curve with BD. Along BD we have

$$R = 980 - 1.4\ (G - 20) = 1008 - 1.4\ G\,.$$

Substitute into the utility function (can also do Lagrange)

$$U = 0.05\ \ln(G) + 0.95\ \ln(1008 - 1.4\, G)$$



The FONC is

$$\frac{dU}{dG} \equiv \frac{0.05}{G} - \frac{0.95 \times 1.4}{1008 - 1.4\,G} = 0\,.$$

This becomes

$$0.05\ (1008 - 1.4\,G) = 1.33\,G, \quad \text{or}|quad 1.4\,G = 50.4, \quad \text{or} \quad G = 36\,.$$

Since $G > 20$, this is a valid solution. Therefore Della buys 36-20 = 16 gallons on the black market.

You can equivalently solve the problem by letting X denote Della's purchase on the black market, and write the budget constraint as

$$20 + 1.4\ X + R = 1000\,.$$

But then you must recognize that her total consumption of gasoline is $(20 + X)$ and maximize the utility

$$0.05\ \ln(20 + X) + 0.95\ R\,.$$

This is the issue pointed out in the Common Errors above.

Extra information: Consider another consumer, Will Walker, whose utility function is

$$W(G, R) = 0.01\ \ln(G) + 0.99\ \ln(R)\,.$$

With only the budget constraint $G + R = 1000$, Will's optimal choice is $G = 10$ and $R = 990$. When rationing is imposed, Will does not even use his full allotment of 20. If you impose $G \leq 20$ as a constraint and do a Kuhn-Tucker solution, the rationing constraint turns out to be slack for Will.

What happens when a black market opens up? If you mechanically use the budget constraint

$$1.4\ G + R = 1008\,,$$

Will's demands can be found from the standard Cobb-Douglas formula, so

$$1.4\ G = 0.01 \times 1008\,, \quad \text{or} \quad G = 10.08/1.4 = 7.2\,.$$

But this number is less than 20. Is this a valid solution? It can be if Will can become a *seller* on the black market. He will take up his full allotment of ration, namely 20 gallons at price 1, and sell $(20 - G)$ at the price of 1.4 enabling him to spend more on other goods. His budget constraint will be

$$R = 1000 - 20 + 1.4\ (20 - G) \quad \text{or} \quad 1.4\ G + R = 1000 - 20 + 28 = 1008\,,$$

the same as when he is a buyer on the black market $(G > 20)$.