ECO 305 - Fall 2003
Microeconomic Theory - A Mathematical Approach
Problem Set 4 - Answer Key

This was graded by Brishti Guha, and the distribution was as follows:

Range	Number
100	4
90–99	67
80–89	8
70–79	2
60–69	2
< 60	3

Question 1: (total 40 points)

COMMON ERRORS: (1) Not getting the economic intuition (dead-weight loss) in part (e). (2) Errors in calculation of the dead-weight loss in part (g). For example, using the Hicksian quantity reduction when calculating the ECO 102 style triangle which is based on Marshallian demand, or calculating the true dead-weight loss as an integral with respect to the Marshallian demand when it should be Hicksian.

(a) (5 points) Marshallian demand functions :

$$F = \frac{M}{2\ P_F} \qquad C = \frac{M}{2\ P_C}\,.$$

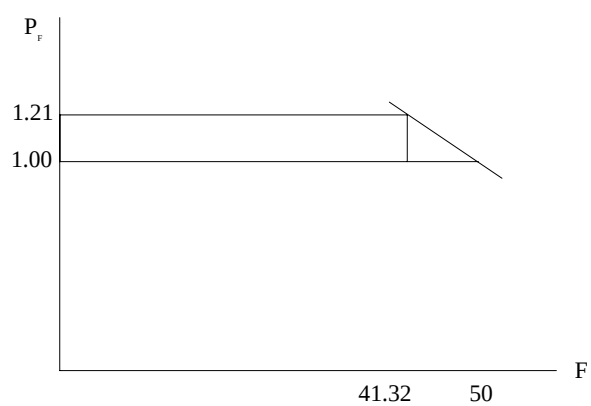
(b) (3 points) $F = 50$, $C = 50$, $U = 100$.
(c) (3 points) $F = 41.32$, $C = 50$, $U = 100/1.1 = 90.91$
(d) (3 points) Tax revenue $R = (1.21 - 1) \times 41.32 = 8.677$.
(e) (6 points) $M^*(1.21, 1, 100) = 1.1 \times 100 = 110$, so need 10 units of extra income to provide (Hicks) compensation.

(3 points) The revenue from the food tax is less, because it casues a dead-weight loss (DWL).

(f) (2 points) Income sufficient to buy old quantities $= 1.21 \times 50 + 1 \times 50 = 110.5$, so Slutsky compensation $= 10.5 >$ Hicks compensation.

(4 points) Given income 110.5, the consumer actually buys $F = 110.5/(2 \times 1.21) = 45.45 < 50$ and $C = 110.5/(2 \times 1) = 55.25 > 50$ and gets utility $110.5/1.1 = 100.45 > 100$.

(g) (3 points) ECO 102 style figure:



(3 points) ECO 102 measure of DWL $= \frac{1}{2}\ (1.21 - 1)(50 - 41.32) = 0.911$

(3 points) The true DWL. It turns out that there are two economically meaningful ways to formulate the concept, and they lead to different numbers. You get full credit for either, so long as you explain what you are doing clearly.

One way to think of dead weight loss is as in part (e). The lump sum that would be needed to restore the consumer to the original utility level exceeds the revenue raised by the tax; the difference is the dead weight loss. Then

$$\text{DWL} = \text{Hicksian compensation needed - Revenue from tax} = 10 - 8.667 = 1.333$$

The other way is to observe that if we could take away more from the consumer in a lump sum than we can using the tax on food, when the amount of the lump sum tax and the rate of tax on food are so chosen as to reduce the consumer to the same utility level, here 90.91. This is the method we used in class. The lump sum amount that could be taken away is

$$M^*(1, 1, 100) - M^*(1, 1, 90.91) = 100 - 90.91 = 9.09$$

Therefore the dead-weight loss is

$$\text{DWL} = \text{Revenue from lump-sum tax - Revenue from food tax} = 9.09 - 8.667 = 0.423$$

(2 points) No matter which concept of DWL you used, it differs from the Marshallian triangle calculation for two reasons: [1] The ECO 102 style measure uses Marshallian demands, not Hicksian ones, and [2] the ECO 102 style measure supposes the demand curve to be a straight line whereas actually it is nonlinear.

Either of these Hicksian concepts answers an economically meaningful question. (The real trouble with the ECO 102 style Marshallian consumer surplus and dead-weight loss concepts is that they don't rigorously measure any economically meaningful magnitude.) Either could be calculated as above, or by calculating an integral with respect to the corresponding Hicksian demand curve (that for utility = 100 in the first way and that for utility = 90.91 in the second way). And either can be approximated by a triangle. You got full credit if you took the triangle approximation and did it correctly. But with the information given, the exact calculation is just as simple to do and of course preferable.

Extra information – however, of the two alternative economically meaningful ways of thinking about the DWL concept, one is less than the Marshallian calculation and the other is greater. So you could think of the Marshallian number as some kind of average of the other two.

Question 2: (total 60 points)

COMMON ERRORS: (1) Leaving out the second-order conditions in the minimization both for K in part (c) and Q in part (d). For one-variable, no-constraint problems you should be able to check SOCs using calculus. For Lagrange problems, we are taking the graphical intuitions. (2) Not many people got the intuition for Q increasing with r in part (d) right. Some of you thought that the Q was the quantity the firm would choose to supply. That can't be right – you can't answer a question about the choice of quantity supplied without knowing the output price, and that was not specified. The Q in this part is a purely technological construct – the quantity that minimizes average cost, or the most cost-efficient scale of production. Some even said that when costs go up, the firm has to produce more to recover the cost. Again in the absence of any information about the output price, this is not true, and in reality such thinking may be a good way to drive a firm into bankruptcy.

(a) (total 15 points) In the short run, to produce Q using $1 + K$ of capital requires $L = Q^6\ K^{-2}$ of labor. therefore (5 points each)

$$\begin{aligned} SRTC(Q;K) &= r\,(1+K) + w\ Q^6\ K^{-2} \\ SRAC(Q;K) &= \frac{r\,(1+K)}{Q} + w\ Q^5\ K^{-2} \\ SRMC(Q;K) &= 6\,w\ Q^5\ K^{-2} \end{aligned}$$

(b) (total 25 points) In the long run the firm solves the problem (10 points for correct solution of this)

$$\text{minimize } r\,K + w\,L \quad \text{subject to} \quad K^{1/3}\,L^{1/6} = Q\,.$$

The Lagrangian conditions for this are

$$r = \lambda\,\frac{1}{3}\,K^{-2/3}\,L^{1/6} \qquad w = \lambda\,\frac{1}{6}\,K^{1/3}\,L^{-5/6}\,.$$

Dividing,

$$\frac{r}{w} = 2\,\frac{L}{K}, \qquad \text{or} \qquad \frac{K}{L} = \frac{2\,w}{r}\,.$$

Then

$$Q = \left(\frac{2\,w}{r}\,L\right)^{1/3} L^{1/6} = \left(\frac{2\,w}{r}\right)^{1/3} L^{1/2}\,.$$

So

$$L = \left(\frac{2\,w}{r}\right)^{-2/3} Q^2 \qquad K = \left(\frac{2\,w}{r}\right)^{1/3} Q^2\,,$$

and

$$r\,K + w\,L = \left[2^{1/3} + 2^{-2/3}\right]\,r^{2/3}\,w^{1/3}\,Q^2 = \frac{3}{2^{2/3}}\,r^{2/3}\,w^{1/3}\,Q^2\,.$$

Therefore (5 points each)

$$\begin{aligned}
LRTC(Q) &= \begin{cases} 0 & \text{if } Q = 0 \\ r + 3\,2^{-2/3}\,r^{2/3}\,w^{1/3}\,Q^2 & \text{if } Q > 0 \end{cases} \\
LRAC(Q) &= \frac{r}{Q} + \frac{3}{2^{2/3}}\,r^{2/3}\,w^{1/3}\,Q \qquad \text{for } Q > 0 \\
LRMC(Q) &= 3\,2^{1/3}\,r^{2/3}\,w^{1/3}\,Q \qquad \text{for } Q > 0
\end{aligned}$$

(c) (10 points) The short-run average cost formula in (a) is

$$SRAC(Q;K) = \frac{r\,(1+K)}{Q} + w\,Q^5\,K^{-2}\,.$$

Fix Q and differentiate with respect to K:

$$\begin{aligned}
SRAC_K(Q;K) &= \frac{r}{Q} - 2\,w\,Q^5\,K^{-3} \\
SRAC_{KK}(Q;K) &= 6\,w\,Q^5\,K^{-4}
\end{aligned}$$

The FONC for minimizing SRAC with respect to K gives

$$K^3 = \left(\frac{2\,w}{r}\right)\,Q^6\,, \qquad \text{or} \qquad K = \left(\frac{2\,w}{r}\right)^{1/3} Q^2\,,$$

and $SRAC_{KK} > 0$ always so this is a global minimum.

Substituting this value of K into the SRAC function gives

$$\begin{aligned}
SRAC(Q) &= \frac{r}{Q} + \frac{r}{Q}\,\left(\frac{2\,w}{r}\right)^{1/3} Q^2 + w\,Q^5\,\left(\frac{2\,w}{r}\right)^{-2/3} Q^{-4} \\
&= \frac{r}{Q} + \left[\,2^{1/3} + 2^{-2/3}\right]\,r^{2/3}\,w^{1/3}\,Q \\
&= \frac{r}{Q} + \frac{3}{2^{2/3}}\,r^{2/3}\,w^{1/3}\,Q\,,
\end{aligned}$$

This is the same as the equation of the long run average cost curve in (b).

(Extra explanation: Mathematically, if Q and A are two variables, and there is a family of curves in (Q, A) space defined by $A = G(Q, K)$ where G is a function and K is an algebraic constant, such that each member of this family of curves corresponds to the graph of $A = -G(Q, K)$ for one value of K, then the envelope of this family can be found by eliminating K between the pair of equations $A = G(Q, K)$ and $G_K(Q, K) = 0$. This is exactly the procedure we carried out here. Thus the economic idaa that the long-run average cost curve is the (lower) envelope of the whole family of short-run average cost curves corresponding to different short-run-fixed levels of K is also valid in the mathematical sense of envelope.)

(d) (total 10 points) To minimize $LRAC(Q)$, we have (7 points for this solution)

$$\begin{aligned} LRAC'(Q) &= -\frac{r}{Q^2} + \frac{3}{2^{2/3}}\, r^{2/3}\, w^{1/3} \\ LRAC''(Q) &= \frac{2\,r}{Q^3} > 0 \end{aligned}$$

Therefore the FONC yields the global min, and it occurs at

$$Q^2 = \frac{2^{2/3}}{3}\, r^{1/3}\, w^{-1/3} \qquad \text{or} \qquad Q = \frac{2^{1/3}}{3^{1/2}}\, r^{1/6}\, w^{-1/6}\,.$$

Why does this Q increase as r increases? Observe that fixed costs are only capital costs, whereas variable costs are a mixture of capital and labor costs. Therefore an increase in r increases the relative importance of fixed costs. This increases the "most-efficient" or "least-average-cost" scale of production. (3 points for the intuition)