ECO 305 – Fall 2003
Microeconomic Theory – A Mathematical Approach
Problem Set 6 – Answer Key

Arnaud Costinot graded Questions 1 and 2 and compiled the totals; his grading is in red ink. Brishti Guha graded question 3; her grading is in black. If after studying the answer key you have any remaining questions, please take them to the appropriate preceptor.

The distribution was as follows:

Range	Number
100	5
90–99	60
80–89	12
70–79	4
< 70	2

Good work, all. With these scores there are very few common errors, but the few are listed below.

Question 1: (20 points)

COMMON ERRORS: (1) As usual, a few forgot to check the SOCs. To restate the rule, for functions of one variable, you should be able to check the calculus SOCs ($F''(x^*) \leq 0$ for SONC, < 0 for SOSC). For functions of two or more variables, and for Lagrange problems, you should be able to offer some brief geometric argument (shape of the function, or the curvature of the level curves of the objective function relative to that of the constraint). Or you can appeal to standard theory - Cobb-Douglas maximization with a straight line constraint, for example. (2) Vague arguments for the price difference, based on size of demand instead of the correct elasticity comparison. (3) The elasticities should be compared at any common price, not just at the equilibrium. (In fact in this problem, the monopolist has zero cost. Therefore he is maximizing revenue. The maximum occurs where the marginal revenue is zero, which is where the price elasticity of demand equals 1 (in numerical value). Therefore both demand elasticities equal 1 at the price-discriminating monopolist's optimum choices of separate prices for the two groups.)

(a) (7 points) The market demand function is

$$Q = 800\,[1 - \ln(p)] + 200\,[2 - \ln(p)] = 1200 - 1000\ \ln(p)$$

Costs being zero, the profit is

$$\Pi = p\,Q = p\ [1200 - 1000\ \ln(p)]$$

To choose p to maximize this, the FONC is

$$d\Pi/dp \equiv 1200 - 1000\ \ln(p) - 1000\ p\ (1/p) = 200 - 1000\ \ln(p) = 0$$

Check the SOC:

$$d^2\Pi/dp^2 \equiv -\ 1000/p < 0$$

Therefore the profit-maximizing price is the solution to the FONC:

$$\ln(p) = 0.2, \qquad \text{or} \qquad p = e^{0.2} \approx 1.22$$

No need to calculate out the exact number.

(b) (5 points for each type)

The market demand function for type-1 consumers is

$$Q_1 = 800\,[1 - \ln(p_1)]$$

The profit from selling to these is

$$\Pi_1 = p_1\, Q_1 = 800\; p_1\; [1 - \ln(p_1)]$$

Then

$$d\Pi_1/dp_1 \equiv 800\; [1 - \ln(p_1) - p_1\; (1/p_1)] = -\; 800\; \ln(p_1)$$

and

$$d^2\Pi_1/d(p_1)^2 = -\; 800/p_1 < 0$$

Therefore the optimal choice is defined by

$$\ln(p_1) = 0, \qquad \text{or} \qquad p_1 = 1$$

The market demand function for type-2 consumers is

$$Q_2 = 200\,[2 - \ln(p_2)]$$

The profit from selling to these is

$$\Pi_2 = p_2\, Q_2 = 200\; p_2\; [2 - \ln(p_2)]$$

Then

$$d\Pi_2/dp_2 \equiv 200\; [2 - \ln(p_2) - p_2\; (1/p_2)] = 200\; [1 - \ln(p_2)]$$

and

$$d^2\Pi_2/d(p_2)^2 = -\; 200/p_2 < 0$$

Therefore the optimal choice is defined by

$$\ln(p_2) = 1, \qquad \text{or} \qquad p_2 = e$$

(c) (3 points) Type-2 customers pay a higher price. Calculate the price elasticities of demand for the two types:

$$\epsilon_1 = -\,\frac{p_1}{Q_1}\,\frac{dQ_1}{dp_1} = -\,\frac{p_1}{Q_1}\,\frac{-\;800}{p_1} = \frac{1}{1 - \ln(p_1)}$$

and

$$\epsilon_2 = -\,\frac{p_2}{Q_2}\,\frac{dQ_2}{dp_2} = -\,\frac{p_2}{Q_2}\,\frac{-\;200}{p_2} = \frac{1}{2 - \ln(p_2)}$$

Therefore at any common price, the demand of type-2 is less elastic. Therefore the firm finds it optimal to charge them the higher price.

Question 2: (40 points)

COMMON ERRORs: (1) In part (e), some students compared the prices and profits of the two firms with each other, whereas the question asked you to compare the (two firms' common) prices under Bertrand competition against the (two firms' common) prices under collusion. (2) Some answers for the intuition of why profits were higher under collusion were tautological: "profits are always higher under collusion". You must say why: "they can never be lower because the colluding firms could always replicate the pricing under Bertrand competition; if the products have any kind of demand interdependency, then colluding firms can do better by recognizing the impact of the price of each product on the profit of the other." (3) For the price comparison (substitute versus complement), there were basically two groups of mistakes (beside those justifying the false results that they obtained in e). First, some students tried to justify the results by saying that firms maximize profits by increasing prices (if $k > 0$) and increasing demand (if $k < 0$), without realizing the same line of reasoning applied in both cases. Secondly, some students tried to justify the results by saying that firms could take control of the business (if $k > 0$) whereas they still had to compete with some third good (if $k < 0$). Unless the "internalization of the externality" between the two firms was discussed in one way or another, 4 points were taken off.

(a) (1 point) The goods are substitutes if $k > 0$ and complements if $k < 0$.

(b) (2 points) The expressions for profits are

$$\begin{aligned}\Pi_1 &= (p_1 - 1)\ (2 - p_1 + k\,p_2)\\ \Pi_2 &= (p_2 - 1)\ (2 + k\,p_1 - p_2)\end{aligned}$$

(c) (15 points) The FONCs for the Bertrand-Nash equilibrium are

$$\begin{aligned}\frac{\partial \Pi_1}{\partial p_1} &\equiv 1 \times (2 - p_1 + k\,p_2) + (p_1 - 1) \times (-1) = 3 - 2\,p_1 + k\,p_2 = 0\\ \frac{\partial \Pi_2}{\partial p_2} &\equiv 1 \times (2 + k\,p_1 - p_2) + (p_2 - 1) \times (-1) = 3 + k\,p_1 - 2\,p_2 = 0\end{aligned}$$

Subtracting the second FONC from the first yields $(2 + k)(p_2 - p_1) = 0$. Since $-1 < k < 1$, $(2 + k) > 1$. Therefore $p_1 = p_2$. Then, from either FONC, we get

$$p_1 = p_2 = \frac{3}{2 - k}$$

From the demand functions,

$$x_1 = x_2 = 2 - (1 - k)\frac{3}{2 - k} = \frac{2(2 - k) - 3(1 - k}{2 - k} = \frac{1 + k}{2 - k}$$

and then

$$\Pi_1 = \Pi_2 = \left[\frac{3}{2 - k} - 1\right]\ \frac{1 + k}{2 - k} = \frac{(1 + k)^2}{(2 - k)^2}\,.$$

(d) (10 points) The joint profit is

$$\Pi_1 + \Pi_2 = (p_1 - 1)\ (2 - p_1 + k\,p_2) + (p_2 - 1)\ (2 + k\,p_1 - p_2)\,.$$

To choose p_1 and p_2 to maximize this, the FONCs are

$$\begin{aligned}1 \times (2 - p_1 + k\,p_2) + (p_1 - 1) \times (-1) + (p_2 - 1) \times k &= (3 - k) - 2\,p_1 + 2\,k\,p_2 = 0\\ (p_1 - 1) \times k + 1 \times (2 + k\,p_1 - p_2) + (p_2 - 1) \times (-1) &= (3 - k) + 2\,k\,p_1 - 2\,p_2 = 0\end{aligned}$$

Subtracting the second FONC from the first gives $2\,(k + 1)\,(p_2 - p_1) = 0$. Since $(k + 1) > 0$, we must have $p_1 = p_2$. Then

$$p_1 = p_2 = \frac{3 - k}{2\,(1 - k)}\,,$$

$$x_1 = x_2 = 2 - (1 - k)\frac{3 - k}{2\,(1 - k)} = 2 - \frac{3 - k}{2} = \frac{1 + k}{2}\,.$$

and

$$\Pi_1 = \Pi_2 = \left[\frac{3 - k}{2\,(1 - k)} - 1\right]\ \frac{1 + k}{2} = \frac{(1 + k)^2}{4\,(1 - k)}$$

(e) (3 points for each comparison, prices and profits) The joint-profit-maximizing price is higher than the Bertrand-Nash price if and only if

$$\frac{3 - k}{2\,(1 - k)} > \frac{3}{2 - k}$$

that is,

$$(3-k)(2-k) > 6(1-k), \quad \text{i.e.} \quad 6 - 5\,k + k^2 > 6 - 6\,k \quad \text{i.e.} \quad k^2 + k > 0, \quad \text{i.e.} \quad k\,(1+k) > 0.$$

Since $1+k > 0$, the inequality is true if and only if $k > 0$. Thus, when the products are substitutes, the joint-profit-maximizing prices are higher than the Bertrand-Nash prices; when the products are complements, the joint-profit-maximizing prices are lower than the Bertrand-Nash prices. With independent products, the two types of prices are equal.

The joint-profit-maximizing profits are higher than the Bertrand-Nash profits if and only if

$$\frac{(1+k)^2}{4\,(1-k)} > \frac{(1+k)^2}{(2-k)^2}$$

that is

$$4\,(1-k) < (2-k)^2, \quad \text{that is} \quad 4 - 4\,k < 4 - 4\,k + k^2$$

which is true whenever $k \neq 0$. Thus joint-profit-maximizing profits are higher than Bertrand-Nash profits whether the products are substitutes or complements. Only in the case of independent products do we get equality.

(f) (2 points for this part) The comparison of profits is obvious: joint-profit-maximizers could always set the same prices as those in the Bertrand-Nash equilibrium. If they choose anything different, that must be because it increases their profit. And they can recognize the mutual effects of prices on demands for this purpose unless the products are independent.

(4 points for this part) The price comparison has to do with the "externality" of one firms' price on the other's profit. This is an ECO 102 type intuition. For example, if firm 1 lowers its price, this hurts firm 2's demand and therefore profit if the two products are substitutes, and helps if they are complements. When firms act independently, each pursuing its own profit, they neglect this externality. But when they collude, they recognize and "internalize" it. When the products are complement, recognizing that lowering the price of each would help the other, they jointly lower prices. When the products are substitutes, recognizing that raising the price of each would help the other, they jointly raise prices.

Additional information: When products are complements, putting them into the hands of one monopolist would lower prices (which is good for consumers) and raise the monopolist's profit (which is good for him) – this is a rare situation where something is good for everyone. When products are substitutes, putting them into the hands of one monopolist would raise prices (which hurts consumers) and raise the monopolist's profit, but we know that the former effect is bigger (the difference is the dead-weight loss).

This is one perspective on the Microsoft versus Netscape story. If operating systems and browsers are complements, then letting Microsoft have a monopoly of both would be good for profits and consumers. If they are substitutes, then a Microsoft monpoly would be socially bad. Of course Microsoft claims the former to be true, and its rivals claim the latter to be true. The government's case was roughly that the two products started out as complements, but would eventually become substitutes as browsers are improved and become able to perform more and more of the functions of operating systems.

Question 3: (40 points)

COMMON ERRORS: (1) In part (d) some people just compared the collusive profit at $p = 3.5$ to the profit in the Bertrand equilibrium at $p = 3$ and stated that as the former was higher they would push the price all the way to 3.5. They did not show or argue that profit was increasing in p throughout the range $(3, 3.5)$ thereby ruling out the possibility that a price like 3.3 could have done better than p=3.5. (2) A less common mistake was that some people did not realize that X shrinks as p goes higher than 3.5.So,they applied the result of the previous case,of profit increasing in p,to case (e)as well and said that price would be pushed all the way to 4.

(a) (8 points) Writing P_j for Jen's price and P_b for Berry's price, a customer located x away from Jen (and therefore $(1-x)$ away from Berry), finds it better to buy from Jen if and only if

$$P_j + x < P_b + (1-x), \qquad \text{or} \qquad x < \frac{1 + P_b - P_j}{2}$$

Therefore the quantities are

$$X_j = \frac{1 + P_b - P_j}{2} \qquad \text{and} \qquad X_b = 1 - X_j = \frac{1 + P_j - P_b}{2}$$

(Because the consumers are modeled as continuously distributed, it makes no difference whether you assign the customer at the exact point of indifference to Jen or Berry.)

(b) (5 points) The profits are

$$\begin{aligned} \Pi_j &= (P_j - 2)\, X_j = \tfrac{1}{2}\,(P_j - 2)\,(1 + P_b - P_j) \\ \Pi_b &= (P_b - 2)\, X_b = \tfrac{1}{2}\,(P_b - 2)\,(1 + P_j - P_b) \end{aligned}$$

(c) (15 points) To get the two best response functions, we have

$$\frac{\partial \Pi_j}{\partial P_j} \equiv \tfrac{1}{2}\,[(1 + P_b - P_j) - (P_j - 2)] = \tfrac{1}{2}\,[3 + P_b - 2\,P_j]$$

and

$$\frac{\partial^2 \Pi_j}{\partial P_j^2} \equiv -\,1 < 0$$

so Jen's best response function is defined by

$$3 + P_b - 2\,P_j = 0, \qquad \text{or} \qquad P_j = (3 + P_b)/2$$

Similarly, Berry's best response function is defined by

$$3 + P_j - 2\,P_b = 0, \qquad \text{or} \qquad P_b = (3 + P_j)/2$$

Solving these simultaneously,

$$P_j = 3/2 + (3 + P_j)/4, \quad \text{so} \quad P_j/4 = 3/2 + 3/4 = 9/3, \quad \text{so} \quad P_j = 3$$

Similarly $P_b = 3$. Then $X_j = X_b = \frac{1}{2}$, and

$$\Pi_j = \Pi_b = \tfrac{1}{2}\,(3 - 2)(1 + 3 - 3) = \tfrac{1}{2}$$

(d) (4 points) If Jen and Berry collusively charge a common price P, the total of price and walking costs that the customer farthest from the kiosks (at $x = \frac{1}{2}$) "pays" is $(P + \frac{1}{2})$. If this is to be no more than 4, we need $P \leq 3.5$. So long as this is true, each sells $X = \frac{1}{2}$ and has profit

$$\Pi = \tfrac{1}{2}\,(P - 2)$$

This is an increasing function of P. Therefore they do want to raise P up to the limit 3.5

(e) (5 points) If the two sellers raise their price even higher, each will sell to X miles of customers defined by $P + X = 4$, or $X = 4 - P$. The profit of each will be

$$\Pi = (P - 2)\,(4 - P) = -\,8 + 6\,P - P^2$$

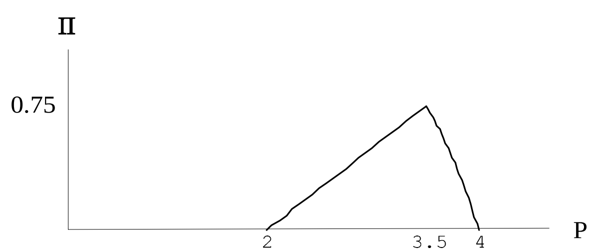
Then

$$d\Pi/dP = 6 - 2\,P$$

which is < 0 when $P > 3.5$. So they do not want to raise the price any higher.

(f) (3 points) Combining the findings in (d) and (e), the optimal collusive price is 3.5, and each sells $\frac{1}{2}$. Therefore the profit of each is $\frac{1}{2}\,(3.5 - 2) = 0.75$.

Additional information: The full graph of the profit of each as a function of their common price P is shown in the figure below; it is an example of a "maximum at a kink" that we saw in the math of the first week.


Π
0.75
P
2
3.5
4