ECO 305 – Fall 2003
Microeconomic Theory – A Mathematical Approach
Problem Set 8 – Due December 11 in class

Disclaimer: Any similarity between the companies and people in this problem set and those in the real world in purely coincidental!

Question 1: (60 points)

Note: You need to be careful with your numerical calculations in this question. If you use a calculator, show your answers to four decimal places. Or you may manipulate and show them as fractions.

The country of Euphoria is about to hold its 2xxx presidential election. The two candidates are Bore and Gush. There are three possible outcomes (scenarios): (1) Bore wins; label subscript B, probability 40%. (2) A cliffhanger; label subscript C, probability 20%. (3) Gush wins; label subscript G, probability 40%.

There are three firms in the economy of Euphoria: (1) Nerds-R-Us, the computer and software company; label subscript N. (2) Root-and-Branch, the petroleum company; label subscript R; (3) Omniscient Eye Network, the multimedia company; label subscript O.

There are three people: (1) Gill Bates, who owns Nerds-R-Us, (2) Hal Burton, who owns Root-and-Branch, and (3) Red Nutter, who owns Omniscient Eye.

(To justify the price-taking behavior that is assumed below, think of each firm and each owner as representing hundreds of identical ones.)

Gush comes from the oil state of Southern Cross; if he is elected, he will adopt policies that favor Root-and-Branch. Bore claims to have invented the Internet; if he wins, he will adopt policies that favor Nerds-R-Us. If there is a cliffhanger, there will be gridlock and the economy will suffer, but everyone will be watching TV so Omniscient Eye will do well. The total market capitalization of the three companies in each scenario (after the fact, when the scenario actually materializes), can be calculated ahead of time. The values, measured in Biggabucks (billions of the Euphoric currency unit, the Buck), are known to be as shown in the following table:

	Scenarios		
Company	Bore wins	Cliffhanger	Gush wins
Nerds-R-Us	3	1	2
Root-and-Branch	2	1	3
Omniscient Eye	1	2	1

(a) Each firm's values in the three scenarios form a three-dimensional vector. There are three such vectors, one for each firm. Are the three vectors linearly independent?

(b) Right now, before the election occurs, markets are held for three Arrow-Debreu securities (ADS) corresponding to the three scenarios. An ADS for scenario i is a contract that will pay the holder, after the outcome of the election is known, 1 Biggabuck if scenario i materializes and nothing if any other scenario materializes. Write P_B for the price in today's trade of one Bore-wins scenario ADS, and similarly P_C and P_G for the prices of the other two ADSs. Each of the three people, through his or her ownership of a company, implicitly owns a set of the three Arrow-Debreu securities. Write down expressions for the values (in today's ADS markets) of the three people's ownership of these securities.

(c) When the three trade ADS's, they will end up choosing different quantities of the ADS's than their initial ownerships. Writing X_B^{Gill}, X_C^{Gill}, and X_G^{Gill} for the quantities chosen by Gill, write down her budget constraint for trade in today's ADS markets.

(d) Suppose each of the three has a logarithmic von Neumann-Morgenstern utility function. When Gill chooses the quantities X_B^{Gill}, X_C^{Gill}, and X_G^{Gill}, what is her final wealth in the three scenarios? Write down the expression for her expected utility. Write similar expressions for Hal and Red.

(e) Find the demand functions of the three people for Arrow-Debreu securities.

(f) In the equilibrium of the markets for the three ADS's, what are their relative prices?

(g) If the election is imminent, so there is insignificant discounting between now and the time when the ADS's pay out the promised sums, and a Buck now is equivalent to a sure Buck after the election, what are the absolute prices of the ADS's?

(h) What does this imply about the prices of the stock of the three firms if these are traded in equity markets before the election? Choose units so that complete ownership of one company is called "one unit of stock," and partial ownership then becomes the appropriate fraction of the stock.

(i) What does this imply about the value, in pre-election markets, of the following derivative securities: (1) A call option on Omniscient Eye that gives you the right, but not the obligation, after the uncertainty is resolved, to buy one unit of its stock (that is, the whole company) for 1.6 Biggabucks (called the strike price of the option). (2) A put option on Root-and-Branch, which gives you the right, but not the obligation, after the uncertainty is resolved, to sell one unit of its stock for 2.4 Biggabucks.

(j) Calculate the quantities X_B^{Gill}, X_C^{Gill}, and X_G^{Gill} of the three ADS's chosen by Gill in the equilibrium of the ADS markets. Do similar calculations for Hal and Red.

(k) Find fractions of stock ownership, written S_N^{Gill}, S_R^{Gill}, and S_O^{Gill} of the three firms that will give her exactly the same final wealth amount in each scenario as she gets from her choice of quantities X_B^{Gill}, X_C^{Gill}, and X_G^{Gill} of the ADS's you found in (j) above. Do the same for Hal and Red. Comment on your results.

(l) What linear combination of the three firms' stocks is equivalent to one ADS for the "Bore wins" scenario?

Question 2: (40 points)

In the country of Toyland, the Dodgem car company has a monopoly of the auto industry. It can produce its Dodgems in two models of different qualities Q, where Q is a continuous and non-negative variable, and charge prices that vary with quality. The cost of producing each auto of quality Q equals Q^2. (This is Q-squared; the 2 is not a superscript label.) There are two types of potential buyers of autos, labelled 1 and 2. The consumer surpluses of the two types, when using a car of quality Q purchased at price P, are given by

$$S_1(P, Q) = 4\,Q - P, \qquad S_2(P, Q) = 6\,Q - P$$

There are N_1 people of type 1 and N_2 people of type 2. Every person will buy exactly one car if he/she gets a non-negative surplus from doing so. If offered a choice between two models $i = 1$, 2 with different (Q_i, P_i), each of which yields non-negative surplus, then he/she will buy the one that yields the biggest surplus. If type-2 customers get equal non-negative surplus from the two models, they buy with the higher quality. If type-1 customers get equal non-negative surplus from the two models, they buy the one with the lower quality.

(a) First suppose that Dodgem can observe the type of each individual. Therefore it can confront every customer of type i with a take-it-or-leave-it offer consisting of quality Q_i and price P_i. The only constraints are the "participation constraints" ensuring non-negative surplus for each type, namely

$$
\begin{aligned}
P_1 &\leq 4\,Q_1 \qquad (1)\\
P_2 &\leq 6\,Q_2 \qquad (2)
\end{aligned}
$$

Show that the firm's profit

$$N_1\,[\,P_1 - (Q_1)^2\,] + N_2\,[\,P_2 - (Q_2)^2\,] \qquad (3)$$

is maximized by choosing

$$Q_1 = 2, \quad P_1 = 8, \qquad Q_2 = 3, \quad P_2 = 18.$$

(b) Now suppose Dodgem cannot recognize the type of any individual. But it knows the total numbers N_1 and N_2 in the population of potential buyers. It devises the following scheme of screening by self-selection. It makes available on the market two models: "quality Q_1, price P_1" (model 1, intended for type 1 customers) and "quality Q_2, price P_2" (model 2, intended for type 2 customers). It must leave the customers free to choose either model. Show that the "self-selection constraints", which ensure that neither type prefers the model intended for the other type, can be written as

$$
\begin{aligned}
4\,(Q_2 - Q_1) &\leq P_2 - P_1 \qquad (4)\\
P_2 - P_1 &\leq 6\,(Q_2 - Q_1) \qquad (5)
\end{aligned}
$$

(c) Show that (4) and (5) together imply $Q_2 \geq Q_1$ and $P_2 \geq P_1$.

(d) Show that if (5) and (1) are satisfied, then (2) is automatically satisfied (so it need not be imposed as a separate constraint in any subsequent maximization).

(e) First consider Dodgem's profit-maximizing choice of prices for any given values of Q_1 and Q_2. It wants prices to be as high as possible, consistent with the constraints. Show that it should set

$$P_1 = 4\,Q_1, \qquad P_2 = 6\,Q_2 - 2\,Q_1$$

(f) Now substitute these expressions for P_1 and P_2 into the expression (3) for Dodgem's profit, and consider its choice of qualities. Show that the optimum choice of Q_2 is $Q_2 = 3$, whereas

$$Q_1 = \begin{cases} 2 - (N_2/N_1) & \text{if } 2N_1 > N_2 \\ 0 & \text{if } 2N_1 \leq N_2 \end{cases}$$

(g) Calculate the numerical values of Dodgem's optimal choices of the qualities and prices when $N_1 = N_2 = 100$. Give the economic intuition why these choices differ from those in (a).

Do the same calculation for $N_1 = 100$ and $N_2 = 300$. What is the intuition for the difference between this answer and that for $N_1 = N_2 = 100$?