ECO 305 – Fall 2003
Microeconomic Theory – A Mathematical Approach
Problem Set 8 – Answer Key

This was graded by Avinash Dixit, and the distribution was asa follows:

Range	Number
90–99	26
80–89	10
70–79	1
< 70	2

Question 1: (60 points)

COMMON ERRORS: (1) Failing to state how some results were derived. It was OK to use a calculator to solve simultaneous linear equations in parts (k) and (l), but you should say that was what you did. If you did matrix inversion or row reduction by hand, you should show some of these steps at least. Merely writing down the answers makes a reader wonder. (2) When valuing options in part (i), some people multiplied the profit from exercising the option in the various scenarios by the probabilities of the scenarios. This is how contingent claims in scenarios would be valued under risk-neutrality (or even if there is only one price-taking risk-neutral trader). But when all traders are risk-averse, that plays a role in determining valuations of scenario-contingent claims (the Arrow-Debreu securities). You have to use the equilibrium prices of these securities as found in part (g). (3) A few people misunderstood the concept of an option, and said that since the whole of Eye is valued at only 1.2727 Biggabucks whereas the exercise price of the option is 1.6 Biggabucks, the option should be worthless. Not so. The price of the company in today's markets reflects the fact that in some scenarios it may make low profits, and in some scenarios high profits. These are valued using the prices of claims to Bucks in the respective scenarios, namely the ADS prices. A call option selectively lets you get the company only in the good scenarios where its profit exceeds the exercise price. So it has positive value, unless the exercise price is higher than the company's profit in every scenario of positive ADS value.

(a) (2 points) The three vectors are $(2, 1, 3)$, $(3, 1, 2)$, $(1, 2, 1)$. Linear independence requires that

$$\det \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \neq 0$$

The determinant equals

$$2 \times 1 \times 1 + 1 \times 2 \times 1 + 3 \times 2 \times 3 - 3 \times 1 \times 1 - 1 \times 1 \times 3 - 2 \times 2 \times 2 = 2 + 2 + 18 - 3 - 3 - 8 = 8 \,.$$

So the condition for linear independence is satisfied.

You can also do this by row reduction etc.

(b) (3 points) Initial ownership values:

$$\begin{array}{ll} \text{Gill} & 3\ P_B + P_C + 2\ P_G \\ \text{Hal} & 2\ P_B + P_C + 3\ P_G \\ \text{Red} & P_B + 2\ P_C + P_G \end{array}$$

(c) (3 points) Budget constraints:

$$\begin{array}{ll} \text{Gill} & P_B\ X_B^{Gill} + P_C\ X_C^{Gill} + P_G\ X_G^{Gill} \leq 3\ P_B + P_C + 2\ P_G \\ \text{Hal} & P_B\ X_B^{Hal} + P_C\ X_C^{Hal} + P_G\ X_G^{Hal} \leq 2\ P_B + P_C + 3\ P_G \\ \text{Red} & P_B\ X_B^{Red} + P_C\ X_C^{Red} + P_G\ X_G^{Red} \leq P_B + 2\ P_C + P_G \end{array}$$

(d) (3 points) Final wealth in any scenario is equal to the chosen holding of the ADS for that scenario. Therefore expected utilities:

$$\begin{array}{ll} \text{Gill} & 0.4\ \ln(X_B^{Gill}) + 0.2\ \ln(X_C^{Gill}) + 0.4\ \ln(X_G^{Gill}) \\ \text{Hal} & 0.4\ \ln(X_B^{Hal}) + 0.2\ \ln(X_C^{Hal}) + 0.4\ \ln(X_G^{Hal}) \\ \text{Red} & 0.4\ \ln(X_B^{Red}) + 0.2\ \ln(X_C^{Red}) + 0.4\ \ln(X_G^{Red}) \end{array}$$

(e) (6 points) Using the standard Cobb-Douglas formula (fine to do this but should mention Cobb-Douglas, not merely write down the answers without any explanation), the demand functions are

$$\begin{array}{llll} \text{Gill} & X_B^{Gill} = 0.4\ \dfrac{3\ P_B + P_C + 2\ P_G}{P_B}, & X_C^{Gill} = 0.2\ \dfrac{3\ P_B + P_C + 2\ P_G}{P_C}, & X_G^{Gill} = 0.4\ \dfrac{3\ P_B + P_C + 2\ P_G}{P_G} \\ \text{Hal} & X_B^{Hal} = 0.4\ \dfrac{2\ P_B + P_C + 3\ P_G}{P_B}, & X_C^{Hal} = 0.2\ \dfrac{2\ P_B + P_C + 3\ P_G}{P_C}, & X_G^{Hal} = 0.4\ \dfrac{2\ P_B + P_C + 3\ P_G}{P_G} \\ \text{Red} & X_B^{Red} = 0.4\ \dfrac{P_B + 2\ P_C + P_G}{P_B}, & X_C^{Red} = 0.2\ \dfrac{P_B + 2\ P_C + P_G}{P_C}, & X_G^{Red} = 0.4\ \dfrac{P_B + 2\ P_C + P_G}{P_B} \end{array}$$

(f) (6 points) Equating the total demand and supply for scenario-B ADS's:

$$0.4\ \frac{3\ P_B + P_C + 2\ P_G}{P_B} + 0.4\ \frac{2\ P_B + P_C + 3\ P_G}{P_B} + 0.4\ \frac{P_B + 2\ P_C + P_G}{P_B} = 2 + 3 + 1\,,$$

or

$$0.4\ \left[\ 6 + 4\ \frac{P_C}{P_B} + 6\ \frac{P_G}{P_B}\ \right] = 6\,,$$

or

$$4\ \frac{P_C}{P_B} + 6\ \frac{P_G}{P_B} = 15 - 6 = 9$$

so

$$-\ 9\ P_B + 4\ P_C + 6\ P_G = 0\,.$$

Similarly equating the demand and supply for scenario-G ADS's, we have

$$6\ P_B + 4\ P_C - 9\ P_G = 0\,.$$

(I have exploited the symmetry between the B and G scenarios to avoid repeating the calculation.) Subtracting one equation from the other yields $15\,(P_B - P_G) = 0$, so $P_B = P_G$, and then $4\,P_C = 3\,P_B$, so $P_C = (3/4)\,P_B = 0.75\,P_B$.

Additional information: (1) By Walras' Law, we need to use the equilibrium conditions in only two of the three markets. You could have used B and C, or C and G. (2) The demand functions are homogeneous of degree zero, therefore the information so far can determine only relative prices. The additional information supplied in part (g) that the three prices sum to 1 enables us to fix the absolute prices. (3) You might wonder why the C-scenario ADS is less valuable than either of the other two, even though there is less total wealth in the C-scenario than in each of the other two (4 as opposed to 6 Biggabucks). The reason is that scenario C is only half as likely as each of the other two (probability 0.2 as opposed to 0.4). Therefore the people are content to make less provision for it. After making this adjustment, scenario C ADS's should be less valued in today's markets.

(g) (2 points) The three ADS's together constitute claim to a sure Biggabuck, and this is valued at 1. (Again you have to state this explanation.) Therefore $P_B + P_C + P_G = 1$, and then

$$P_B = \frac{4}{11} = 0.3636, \quad P_C = \frac{3}{11} = 0.2727, \quad P_G = \frac{4}{11} = 0.3636$$

(h) (3 points) The stock values of the firms are

$$\begin{array}{rl}
\text{Nerds-R-us} & 3\ \dfrac{4}{11} + 1\ \dfrac{3}{11} + 2\ \dfrac{4}{11} = \dfrac{23}{11} = 2.0909 \\
\text{Root-and-Branch} & 2\ \dfrac{4}{11} + 1\ \dfrac{3}{11} + 3\ \dfrac{4}{11} = \dfrac{23}{11} = 2.0909 \\
\text{Omniscient Eye} & 1\ \dfrac{4}{11} + 2\ \dfrac{3}{11} + 1\ \dfrac{4}{11} = \dfrac{14}{11} = 1.2727
\end{array}$$

(i):(1) (3 points) The value of Omniscient Eye in the scenarios B, C and G is 1, 2 and 1 respectively. Your option lets you buy it at 1.6. Therefore you will not exercise your right (let the option lapse) if scenarios B or G materialize, but if C materializes, you will exercise it and thereby make a profit of 0.4 . Thus, before uncertainty is resolved, your option is exactly like holding 0.4 of the C-scenario ADS. Its value in the pre-election market must be $0.4 \times 3/11 = 0.1091$.

(i):(2) (3 points) The value of Root-and-Branch in the scenarios B, C and G is 2, 1 and 3 respectively. Your option lets you sell it at 2.4. Therefore you will not exercise your right (let the option lapse) if scenario G materializes. You will exercise it if either of the other two scenarios materialize, making a profit of $2.4 - 2 = 0.4$ in B, and $2.4 - 1 = 1.4$ in C . Thus, before uncertainty is resolved, your option is exactly like holding 0.4 of the B-scenario ADS and 1.4 of the C-scenario ADS. Its value in the pre-election market must be

$$0.4 \times \frac{4}{11} + 1.4 \times \frac{3}{11} = \frac{5.8}{11} = 0.5273$$

(j) (9 points) Using the equilibrium prices of the ADS's, we have the initial ownership values

$$\begin{array}{cl}
\text{Gill} & (3 \times 4 + 3 + 2 \times 4)/11 = 23/11 \\
\text{Hal} & (2 \times 4 + 3 + 3 \times 4)/11 = 23/11 \\
\text{Red} & (4 + 2 \times 3 + 4)/11 = 14/11
\end{array}$$

Using these in the demand functions, we have the ADS holding quantitites:

$$\begin{array}{cccc}
\text{Gill} & X_B^{Gill} = 0.4\ \dfrac{23/11}{4/11} = 2.300, & X_C^{Gill} = 0.2\ \dfrac{23/11}{3/11} = 1.5333, & X_G^{Gill} = 0.4\ \dfrac{23/11}{4/11} = 2.300 \\
\text{Hal} & X_B^{Hal} = 0.4\ \dfrac{23/11}{4/11} = 2.300, & X_C^{Hal} = 0.2\ \dfrac{23/11}{3/11} = 1.5333, & X_G^{Hal} = 0.4\ \dfrac{23/11}{4/11} = 2.300 \\
\text{Red} & X_B^{Red} = 0.4\ \dfrac{14/11}{4/11} = 1.400, & X_C^{Red} = 0.2\ \dfrac{14/11}{3/11} = 0.9333, & X_G^{Red} = 0.4\ \dfrac{14/11}{4/11} = 1.400
\end{array}$$

(k) (12 points) If Gill holds fraction S_N^{Gill} of Nerds-R-us stock, S_R^{Gill} of Root-and-Branch stock, and S_O^{Gill} of Omniscient Eye stock, this is just like holding

$$\begin{array}{ll}
3\ S_N^{Gill} + 1\ S_O^{Gill} + 2\ S_R^{Gill} & B\text{-ADS's} \\
1\ S_N^{Gill} + 2\ S_O^{Gill} + 1\ S_R^{Gill} & C\text{-ADS's} \\
2\ S_N^{Gill} + 1\ S_O^{Gill} + 3\ S_R^{Gill} & G\text{-ADS's}
\end{array}$$

For this to replicate Gill's equilibrium ADS holdings,

$$\begin{array}{rcl}
3\ S_N^{Gill} + 1\ S_O^{Gill} + 2\ S_R^{Gill} & = & 2.300 \\
1\ S_N^{Gill} + 2\ S_O^{Gill} + 1\ S_R^{Gill} & = & 1.533 \\
2\ S_N^{Gill} + 1\ S_O^{Gill} + 3\ S_R^{Gill} & = & 2.300
\end{array}$$

We can solve this system of three simultaneous linear equations in three unknowns because the determinant of the coefficient matrix on the left hand sides is non-zero (the linear independence property proved in (a) above). In fact you can solve it by inverting the matrix; Here is a more pedestrian method.

The first and the third equation together imply $S_N^{Gill} = S_R^{Gill}$. Therefore

$$\begin{aligned} 5\ S_N^{Gill} + 1\ S_O^{Gill} &= 2.300 \\ 2\ S_N^{Gill} + 2\ S_O^{Gill} &= 1.533 \end{aligned}$$

Multiply the second of these by 0.5 and subtract from the firt to get

$$4\ S_N^{Gill} = 2.300 - 0.766 = 1.533, \quad \text{therefore} \quad S_N^{Gill} = 0.3833.$$

Then $S_R^{Gill} = 0.3833$, and

$$S_O^{Gill} = 2.300 - 5 \times 0.3833 = 0.3833$$

Thus Gill ends up owning 0.3833 of each of the three firms.

You can do similar calculations for the others, and will find that Hal owns 0.3833 of each firm also (this follows by noticing the symmetry between Hal and Gill on interchanging the Bore and Gush scenarios), whereas poor Red owns only 0.2333 of each (1 minus 2×0.3833). The economically important point to note is that the three risk-averse people are able to diversify away all of their individual risk, and here since they have equal (and constant relative) risk aversion, they end up bearing the aggregate risk equally.

(l) (5 points) Suppose amounts Y_N, Y_O, and Y_R of the three firms' stocks are equivalent to 1 Bore-wins scenario ADS. Equating the payoffs from this portfolio and the ADS in each scenario:

$$\begin{aligned} 3\ Y_N + 2\ Y_R + Y_O &= 1 \\ Y_N + Y_R + 2\ Y_O &= 0 \\ 2\ Y_N + 3\ Y_R + Y_O &= 0 \end{aligned}$$

Subtracting the third from the first, $Y_N - Y_R = 1$. Multiplying the first by 2 and substracting the second from that, $5\ Y_N + 3\ Y_R = 2$. Then multiply the first of these two equations in (Y_N, Y_R) by 3 and add to the second, yielding $8\ Y_N = 5$, so $Y_N = 5/8 = 0.625$. Then $Y_R = -3/8 = -0.375$. Finally,

$$Y_O = -2 \times 5/8 - 3 \times (-3/8) = -1/8 = -0.125$$

Negative amounts correspond to short sales. Thus one Bore-wins ADS can be replicated by holding 5/8 of Nerds-R-Us, and selling short 3/8 of Root-and-Branch and 1/8 of Omniscient Eye.

Bonus information: I asked for three identifications (the rest are obvious). I gave one point for getting two of them, and two points for all three. (1) Euphoria - Although it looks like the U.S., a more precise identification (central California in David Lodge's novel *Changing Places*) was needed. (2) Root-and-Branch: Had to go as far back as Brown and Root, or at least Kellogg, Brown and Root. Not enough to say Halliburton. (3) Red Nutter - Ted Turner. Most of you got only the third.

Important additional information: In the Doc-Geek question of Problem Set 7, I said that justifying the assumption that both of them were price-takers was problematic because if Geek was to stand as a representative of n Geeks, the scenarios where their dotcoms succeeded or failed would have to be perfectly correlated, else there would be 2^n different scenarios. That would make it a very difficult problem. Thre is no similar difficulty here. Gill, Hall, and Red can stand as representatives of several people, each of whom initially owns a small fraction of one of the companies. Since Bore and Gush are not representatives of several candidates, there are only two candidates in the election, and therefore (including the cliffhanger) only three scenarios.

Question 2: (40 points)

COMMON ERRORS: (1) In part (f), since a corner solution $Q_1 = 0$ is possible, you cannot mechanically use a first-order condition $\partial\Pi/\partial Q_1 = 0$. You have to use an inequality condition $\partial\Pi/\partial Q_1 \leq 0$, with equality only when $Q_1 > 0$. Since this was emphasized when we did constrained optimization, I was quite strict in enforcing this. (2) A few people used $P_1 = 4\,Q_1$ in part (d). Actually that does not happen until profit-maximization is considered in part (e). In part (d) we have to consider all feasible prices and qualities.

(a) (8 points) When Dodgem can identify types, to maximize its profit, which is an increasing function of P_1 and P_2, it should charge each type the highest feasible price, thereby reducing his surplus to zero. (Not enough to say "It is optimal for Dodgem to set surpluses of both types equal to zero." Dodgem does not care about consumer surpluses *as such*. The link with profits should be stated.) Therefore

$$P_1 = 4\ Q_1, \qquad P_2 = 6\ Q_2$$

and then

$$\begin{aligned} \pi &= N_1 \,[\, P_1 - (Q_1)^2 \,] + N_2 \,[\, P_2 - (Q_2)^2 \,] \\ &= N_1 \,[\, 4\, Q_1 - (Q_1)^2 \,] + N_2 \,[\, 6\, Q_2 - (Q_2)^2 \,] \end{aligned}$$

The FONCs for the choice of Q_i are

$$N_1\ (4 - 2\, Q_1) = 0, \qquad N_2\ (6 - 2\, Q_2) = 0$$

and the SOSC's are satisfied as the function is concave and constraints linear. Therefore

$$Q_1 = 2,\ Q_2 = 3, \qquad \text{and then } P_1 = 8,\ P_2 = 18$$

(b) (5 points) With individuals not identifiable, the self-selection constraints are

$$4\, Q_1 - P_1 \geq 4\, Q_2 - P_2, \qquad 6\, Q_1 - P_1 \leq 6\, Q_2 - P_2$$

Note how the tie-breaking rules for the two types come into play here. The two can be combined into one pair of inequalities:

$$4\ (Q_2 - Q_1) \leq P_2 - P_1 \leq 6\ (Q_2 - Q_1)$$

which are the (1) and (5) you were asked to prove.

(c) (5 points) These together imply

$$4\ (Q_2 - Q_1) \leq 6\ (Q_2 - Q_1), \qquad \text{or} \qquad (6 - 4)\ Q_1 \leq (6 - 4)\ Q_2$$

Therefore $Q_2 \geq Q_1$. Then

$$P_2 - P_1 \geq 4\,(Q_2 - Q_1) \geq 0, \qquad \text{or} \qquad P_2 \geq P_1$$

(d) (3 points) We have

$$\begin{aligned} P_2 &\leq P_1 + 6\ (Q_2 - Q_1) && \text{by (5)} \\ &\leq 4\ Q_1 + 4\ (Q_2 - Q_1) && \text{by (1)} \\ &= 6\ Q_2 - 2\ Q_1 \leq 6\ Q_2 \end{aligned}$$

which proves (2).

(e) (4 points) Given Q_1 and Q_2, the largest P_1 consistent with (1) is $P_1 = 4\ Q_1$. Then the largest P_2 consistent with (5) is

$$P_2 = P_1 + 6\,(Q_2 - Q_1) = 4\, Q_1 + 6\,(Q_2 - Q_1) = 6\, Q_2 - 2\, Q_1$$

With $P_2 - P_1 = 6(Q_2 - Q_1)$, (4) is automatically satisfied.

(f) (8 points) Substituting these into the expression for profit,

$$\pi = N_1\ [\, 4\, Q_1 - (Q_1)^2 \,] + N_2\ [\, 6\, Q_2 - 2\, Q_1 - (Q_2)^2 \,]$$

Differentiating,

$$\partial\pi/\partial Q_1 = 4\, N_1 - 2\, N_2 - 2\, N_1\ Q_1$$

and

$$\partial\pi/\partial Q_2 = N_2\,[\,6 - 2\,Q_2\,]$$

The SOSCs are met as the function is concave. The FONC with respect to Q_2 is simple and gives an interior solution $Q_2 = 3$. The FONC with respect to Q_1 can however give a corner solution:

$$\text{if } N_2 \geq 2\,N_1, \text{ then } \partial\pi/\partial Q_1 \leq 0 \text{ at } Q_1 = 0$$

Otherwise we have the interior solution to $\partial\pi/\partial Q_1 = 0$,

$$Q_1 = (2\,N_1 - N_2)/N_1$$

(g) (3 points) When $N_1 = N_2 = 100$, so $2\,N_1 > N_2$, we have

$$Q_1 = 1, \; Q_2 = 3, \qquad \text{and then } P_1 = 4, \; P_2 = 16$$

The difference between this and the calculations in (a) and arises because now Dodgem needs to cope with its information limitation. If it charged the full prices extracting all consumer surplus from the two types, then type 2 customers would get surplus $6 \times 3 - 18 = 0$ from model 2, whereas they could have gotten $6 \times 2 - 8 = 4$ from model 1, so they would go for model 1 as well. Then Dodgem would make a profit of only $8 - 2^2 = 4$ from them. It can keep them buying model-2 cars only by lowering the price of these sufficiently to meet the self-selection constraint. But that also reduces its profit. It can stem this reduction to some extent, by lowering Q_1 and thereby making model 1 less attractive for the type 2 customers, who value quality more highly.

The price-lowering intuition is standard, as in the air fares example in class. But the intuition why Q_1 lowered is more subtle. I gave full credit so long as there was a clear idea of screening by self-selection and therefore the constraint of having to accept a lower price from type 2's.

(3 points) When $N_1 = 100$ and $N_2 = 300$, so $2\,N_1 < N_2$, we have

$$Q_1 = 0, \; Q_2 = 3, \qquad \text{and then } P_1 = 0, \; P_2 = 18$$

When type 2 customers are a sufficiently large fraction of the population, the reduction in profit from offering them a price below 18 for model 2 becomes too large to bear. The firm does better by not producing model 1 at all (which is what $Q_1 = 0$ amounts to), even though this means forgoing all profit from type 1 customers. Some people said that there are so few type 1's that the firm cannot make a profit by serving them. Not true; rather, serving them has a knock-on effect on the prices and therefore the profits that can be had from type 2's. That is the key. Again similar to the air fares example in class.