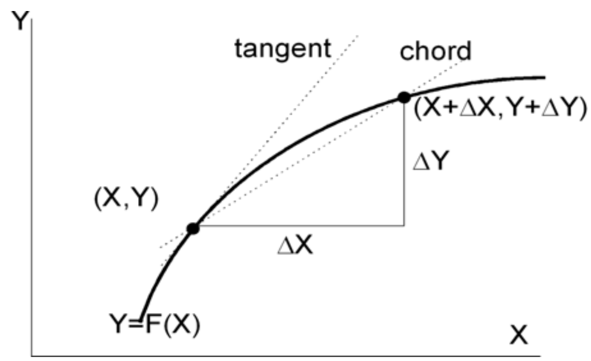
Precepts Week 1: September 20, 21

Review of elasticities

Consider a functional relationship between two economic variable such as outputs and inputs:

$$Y = F(X)$$

Here Y is the dependent variable and X the independent variable. For concretness, suppose X is an input like labor, and Y is an output, so F is a production function. (Other examples can be where X is a price or an income, and Y is the quantity of a good demanded.) In almost all of our applications, X, Y will be positive.



The figure shows the graph of this function, with two adjacent points marked:

$$(X, Y) \qquad \text{and} \qquad (X + \Delta X,\, Y + \Delta Y)$$

So when starting at X the independent variable (quantity of input)increases by ΔX, this causes the dependent variable (quantity of output) to increase by ΔY. The "finite incremental product" is given by the slope of the chord joining these points, $\Delta Y/\Delta X$
In the limit, as we consider infinitesimally small increments, the "marginal product" is the slope of the tangent to the graph at X, or in calculus notation,

$$\text{the derivative } \frac{dY}{dX}, \text{ also written } F'(X),\ F_X(X)$$

Corresponding concepts for proportional changes:

$$\text{Arc elasticity} = \frac{\Delta Y/Y_m}{\Delta X/X_m} = \frac{X_m}{Y_m}\,\frac{\Delta Y}{\Delta X}$$

where $X_m = X + \frac{1}{2}\Delta X$, $Y_m = Y + \frac{1}{2}\Delta Y$ (midpoint of arc)

$$\text{Point elasticity} = \frac{X}{Y}\,\frac{dY}{dX}$$

Example: If $Y = X^{2/3}$, $dY/dX = \frac{2}{3}\ X^{2/3-1} = \frac{2}{3}\ X^{-1/3}$, and

$$\text{elasticity} = \frac{X}{X^{2/3}}\ \frac{2}{3}\ X^{-1/3} = \frac{2}{3}$$

Usefulness of this: under perfect competition, profit-maximizing firms hire labor to the point where its marginal product equals the real wage (wage measured in units of the product). So

$$\frac{W}{P} = \frac{dY}{dX}, \quad \text{and} \quad \frac{W\,X}{P\,Y} = \frac{X}{Y}\ \frac{dY}{dX}$$

that is, the share of wages in the value of output equals the elasticity of the production function.

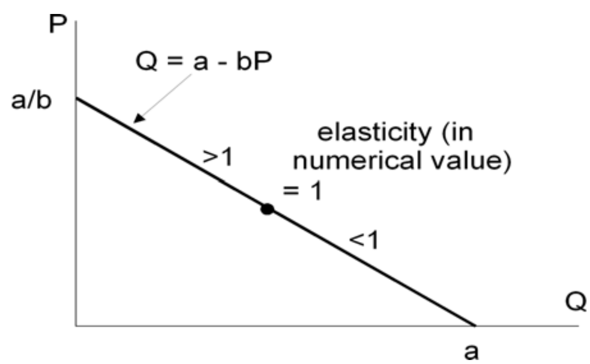
More generally, if $Y = X^k$, then $dY/dX = k\ X^{k-1}$, and

$$\text{elasticity} = \frac{X}{X^k}\ k\ X^{k-1} = k$$

Example: Linear demand curve $Q = a - b\,P$. The price is the independent variable and the quantity the dependent variable, but the unfortunate economics convention shows P on the vertical axis and Q on the horizontal axis. The line meets the horizontal axis ($P = 0$) when $Q = a$ and the vertical axis ($Q = 0$) when $P = a/b$. And the line is downward-sloping (negative slope). So the slope is

$$-\frac{a/b}{a} = -\frac{1}{b}$$

or $1/b$ in numerical value.



The derivative of the demand function is $-b$. So the derviative is the inverse of the slope of the graph (because of the conventional reversal of the axes).

The elasticity of demand is

$$\frac{P}{Q}\ \frac{dQ}{dP} = -\frac{b\,P}{a - bP}$$

Demand is price-elastic (elasticity > 1 in numerical value, when

$$\frac{b\,P}{a - b\,P} > 1, \text{ or } b\,P > a - b\,P,, \text{ or } P > \frac{1}{2}\ \frac{a}{b}$$

So demand is price-elastic along the upper half of the line, and price-inelastic (elasticity < 1 in numerical value) along the lower half of the line.

How much land can a person enclose?

Change the example in the class by supposing that it is harder to run in the x-direction than in the y-direction. Specifically, suppose that it takes one unit of time (7 minutes, say) to run a unit distance in the y-direction (1 mile, say), but it takes k units of time to run a unit distance in the x-direction. If the total time available is T, then the runner's choice of x and y for the sides of the rectangle must satisfy the constraint

$$k\,(2\,x) + 2\,y = T\,.$$

He wants to maximize $S = x\,y$. Use the MAT 102-3 method.

$$S = x\left(\frac{T}{2} - k\,x\right) = \frac{T}{2}\,x - k\,x^2$$

Therefore

$$\frac{dS}{dx} = \frac{T}{2} - 2\,k\,x$$

Setting this equal to zero and solving,

$$x = \frac{T}{4\,k}, \text{ then } y = \frac{T}{4} \text{ and } S = \frac{T^2}{16\,k}$$

In the ECO 100 approach, this is as if the "price" of x has gone up and therefore the "quantity of x demanded" has gone down. The figure shows this using the same numbers as in the class, and $k = 2$.