ECO 300 – MICROECONOMIC THEORY – FALL 2005
SOLUTIONS FOR PRECEPTS WEEK 4 – OCTOBER 19

QUESTION 1:

To min r K + w L subject to $Q = (3/2) (K^{2/3} + L^{2/3})$, the Lagrangian is

$$Z = r K + w L - \lambda [(3/2) (K^{2/3} + L^{2/3}) - Q]$$

The first-order conditions are

$$r - \lambda * (3/2) * (2/3) K^{-1/3} = 0, \qquad w - \lambda * (3/2) * (2/3) L^{-1/3} = 0$$

or

$$K = (\lambda / r)^3 , \ L = (\lambda / w)^3 .$$

Then

$$Q = (3/2) [(\lambda / r)^2 + (\lambda / w)^2] = (3/2) \lambda^2 [r^{-2} + w^{-2}]$$

Therefore

$$\lambda = (2/3)^{1/2} Q^{1/2} [r^{-2} + w^{-2}]^{-1/2}$$

So

$$K = (2/3)^{3/2} Q^{3/2} [r^{-2} + w^{-2}]^{-3/2} r^{-3} , \ L = (2/3)^{3/2} Q^{3/2} [r^{-2} + w^{-2}]^{-3/2} w^{-3}$$

Finally

$$\begin{aligned} C &= (2/3)^{3/2} Q^{3/2} [r^{-2} + w^{-2}]^{-3/2} r^{-3} r + (2/3)^{3/2} Q^{3/2} [r^{-2} + w^{-2}]^{-3/2} w^{-3} w \\ &= (2/3)^{3/2} Q^{3/2} [r^{-2} + w^{-2}]^{-3/2} [r^{-2} + w^{-2}] \\ &= (2/3)^{3/2} Q^{3/2} [r^{-2} + w^{-2}]^{-1/2} \end{aligned}$$

If K and L both increase by a factor s > 1, output becomes

$$(3/2) [(s K)^{2/3} + (s L)^{2/3}] = (3/2) s^{2/3} [(K)^{2/3} + (L)^{2/3}] ,$$

that is, output increases by the factor $s^{2/3}$. This is < s when s > 1. Therefore there are diminishing returns to scale.

Total cost is proportional to $Q^{3/2}$. Therefore average cost is proportional to $Q^{1/2}$. That is, average cost is increasing as Q increases.

QUESTION 2:

(a) The volume of a tank goes up as the cube of its linear dimension, whereas the metal required to make it is proportional to its surface area, which goes up only as the square of its linear dimension. Therefore the cost goes up as the 2/3 power of the quantity. For any aficionados of Lagrange’s method among you, we offer details at the end of this handout.

That one person can operate a tank regardless of its size is a fact of many modern technological processes.

The cost of selling goes up as the square of the quantity, that is, the marginal cost increases. For small quantities, you can just rely on repeat orders from your regular loyal customers; perhaps all that is needed is a little reinforcement/reminder advertising or a friendly visit from your sales staff to the stores that buy and store your product to maintain relations. For larger quantities you have to compete harder to win or keep less loyal buyers or compete harder with rival firms' salespeople.

(b) The total purchase cost of a tank volume 8 is $= k\ 8^{2/3} = 4\ k$. The part attributed to each year of the tank's 4-year life is therefore k. This one tank suffices so long as $Q < 8$. In this range, therefore total cost $= k + w + c\ Q^2$

Of this, k is sunk and $w + c\ Q^2$ is avoidable; $k + w$ is fixed and $c\ Q^2$ is variable. The operator of the tank is hired afresh each year so his/her salary cost is not sunk; if for any reason the firm decides to produce nothing in the coming year ($Q = 0$), it can choose not to hire the operator.

If $Q > 8$, the firm has to buy another tank or tanks. There are economies of scale in the production and operation of a tank. Therefore we need to consider two possibilities:

(i) The firm buys just one new tank of capacity (Q-8). The cost of this, over each year of its life, is $(k/4)\ (Q\text{-}8)^{2/3}$. Then it will need two people to operate the two tanks. Therefore the coming year's total cost is $= k + (k/4)\ (Q\text{-}8)^{2/3} + 2\ w + c\ Q^2$

Here only k is sunk; the rest is avoidable. However, if the firm chooses this path, next year the $(k/4)\ (Q\text{-}8)^{2/3}$ will also be sunk. When in the future the old tank comes to the end of its life, the k will cease to be sunk.

Alternatively, the firm can abandon the existing tank (but must still go on bearing its sunk cost) and buy a new huge tank that can produce all of Q. This saves the cost of hiring one operator. Then total cost $= k + (k/4)\ Q^{2/3} + 2\ w + c\ Q^2$

Again only k is sunk; in the following year the $(k/4)\ Q^{2/3}$ will also be sunk, and eventually the k will cease to be sunk.

The firm will choose the first or the second of these options according to whether

$$(k/4)\ (Q\text{-}8)^{2/3} + 2\ w \ < \text{ or } >\ (k/4)\ Q^{2/3}$$

(c) If $Q < 8$,

$MC = 2\ c\ Q$
$AC = (k+w)/Q + c\ Q$
$AVC = c\ Q$
$AAC = w/Q + c\ Q$

The expressions if $Q > 8$ depend on which option gets chosen. They follow from routine differentiation but are messier; we leave it for you to work them out.

Here are the details of the relation between the cost and the volume of a tank. If the cylinder has radius R and height H, the volume is $V = \pi R^2 H$. The surface area comprises the top and the bottom which are two discs of radius R each, and the side which has height H and circumference $2 \pi R$. Therefore the surface area is $S = 2 \pi R^2 + 2 \pi R H = 2 \pi (R^2 + R H)$. To construct a tank of a given volume using the minimum amount of metal, therefore, the producer of a tank should choose R and H to minimize $R^2 + R H$ subject to the constraint $V = \pi R^2 H$, or $R^2 H = V / \pi$. The Lagrangian for this is

$$L = R^2 + R H - \lambda [R^2 H - V / \pi]$$

The first-order conditions are $2 R + H - 2 \lambda R H = 0$, $R - \lambda R^2 = 0$. Then $\lambda = 1/R$, and $2 R + H - 2 H = 0$, or $2 R = H$.

Using this in the constraint, $2 R^3 = V / \pi$, or $R = [V / (2\pi)]^{1/3}$. Then the minimized surface area is

$$S = 2 \pi R^2 + 2 \pi R H = 2 \pi R^2 + 2 \pi R * 2 R = 6\pi R^2 = 6\pi [V / (2\pi)]^{2/3}$$

Therefore the amount of the metal required is proportional to $V^{2/3}$.

This is only a simple mathematical pass at the problem. In practice, considerations of chemistry of the process or engineering may require the tank designer to depart from the surface-minimizing configuration of $H = 2 R$. This won’t by itself change the 2/3 power result. A larger tank will have to be thicker; that will change the result but we expect the power to be somewhere between 2/3 and 1. Other considerations such as the sizes of the inlet and outlet tubes are likely to have smaller effects.