A Cookie-Cutting Problem*

by
Avinash Dixit
Princeton University

This version: February 22, 2021

Toni likes black-and-white cookies.[1] Being health-conscious, she allows herself only a quarter of one cookie at a time. She wants equal parts of the chocolate and vanilla coatings. And she wants to achieve this with a single cut, perpendicular to the diameter separating the chocolate and the vanilla halves (see Figure 1). How far in from the edge of the cookie should she make the cut?

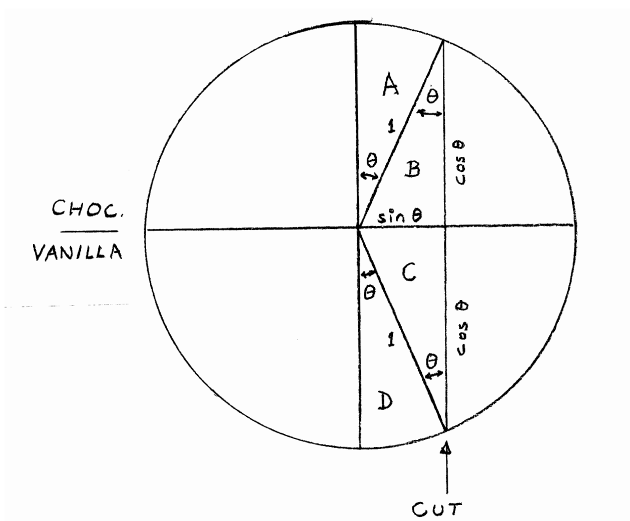

Figure 1: Cookie showing halves and cut

I model this in two ways. First, I regard the cookie as a flat disc with equal thickness throughout. Actually the cookie is thickest at the center and gradually tapers toward the

* I thank Toni Adlerman for suggesting this problem, and James and Timothy Poterba for simplifying suggestions on an earlier draft.

[1] Explanation for non-Americans: one half is covered with chocolate icing, the other half vanilla icing.

edges, so a better approximation may be to regard it as an ellipsoid. That is my second model.

Disc model

This is the one shown in Figure 1. Without loss of generality (by choice of units), take the radius of the cookie to equal 1. Then the area of the disc is π, and with constant thickness, this can also stand for the volume of the cookie. Suppose the radii from the ends of the cut subtend an angle θ with the cut. We want the area to the right of the cut to equal $\pi/4$. Therefore the area between the cut and and the vertical line through the center of the disc[2] should also equal $\pi/4$.

The area has been divided into four parts, labeled A to D. The area of each can be calculated very easily.

Portion A is a wedge of angle θ, so its area is $\theta/(2\pi)$ times the area of the whole disc, or $\theta/2$. Portion D has the same area, so the two together make θ.

Portion B is a right-angled triangle with the horizontal side equal to $\sin(\theta)$ and the vertical side equal to $\cos(\theta)$, so its area is $\frac{1}{2}\sin(\theta)\cos(\theta)$. Portion C has the same area, so the two together make $\sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$. (See Appendix B.)

We want the four portions to add up to a quarter of the area of the disc, namely $\pi/4$. So the equation defining θ is

$$\theta + \tfrac{1}{2}\sin(2\theta) = \pi/4$$

The derivative of the left hand side is $1 + \cos(2\theta) > 0$, and the left hand side is 0 when $\theta = 0$ and $\pi/2$ when $\theta = \pi/2$. Therefore the equation has a unique solution for θ.

Tabulating the left hand side (easily done using Excel or some such program), we find $\theta \approx 0.41$ radians and $\sin(\theta) \approx 0.4$. Therefore the cut should be about 40% away from the center toward the edge, or 60% of the way in from the edge.

Ellipsoid model

Now regard Figure 1 as showing the flat face of the cookie and take its radius to equal 1. Let the horizontal axis (the line separating the chocolate and vanilla halves) be the x-axis, and

[2] To get her second quarter-cookie the next day, Toni will make her second cut along this vertical line through the center. Unless Avinash beats her to it :-)

the vertical axis the y-axis. The thickness goes along the z-axis pointing toward the viewer. Let the thickness be $2t$. Then the equation of the surface of the ellipsoid is

$$x^2 + y^2 + (z/t)^2 = 1$$

The volume of the ellipsoid is $(4/3) * \pi * 1 * 1 * t = 4\pi t/3$.

If a cut is made at x, the cross-section exposed is an ellipse with equation

$$y^2 + (z/t)^2 = 1 - x^2$$

So its largest and smallest radii have lengths $(1-x^2)^{1/2}$ and $t\,(1-x^2)^{1/2}$. Therefore its area is $\pi\, t\,(1-x^2)$.

Integrating such cross-sections, the volume to the right of a cut made at $x = k$ is

$$\int_k^1 \pi\, t\,(1-x^2)\,dx = \pi\, t \left[\, (1-k) - \frac{1}{3}(1-k^3) \,\right] = \pi\, t \left[\, \frac{2}{3} - k + \frac{1}{3}k^3 \,\right]$$

We want this to equal 1/4 of the volume of the ellipsoid. Therefore

$$\pi\, t \left[\, \frac{2}{3} - k + \frac{1}{3}k^3 \,\right] = \pi\, t/3$$

or

$$\frac{2}{3} - k + \frac{1}{3}k^3 = 1/3$$

or

$$1 - 3k + k^3 = 0$$

The derivative of the left hand side is $3\,(k^2-1)$, which is negative over the range $k \in (0,1)$. The left hand side equals 1 at $k = 0$ and -1 at $k = 1$. Therefore the equation has a unique solution. Using Excel, we find that $k \approx 0.35$. So the cut should be made about 35% away from the center, or about 65% in from the edge.[3]

This accords with intuition – with thinner edges playing a bigger role in the portion to the right of the cut in the figure, the cut should go somewhat deeper than in the calculation that assumed equal thickness throughout. Luckily the difference is small, so a cut somewhere in that range, say a little less than 65% in from the edge, should be good enough in practice.

Readers who want a fraction other than a quarter can easily adapt these models. The trigonometry and the calculus parts are unaltered; it only remains to substitute the desired proportion instead of 1/4 in deriving the final equations defining the cut, and re-do the Excel calculations.

[3] Observe that the answer is independent of the thickness t. So (fortunately) I didn't have to ruin a cookie trying to measure t using a Vernier caliper or some such instrument!

Appendix

A: Disc model using calculus

Trigonometry sufficed for the Disc model, but we can also use calculus as we did for the Ellipsoid model. The height of the vertical cut at x is $2\,(1-x^2)^{1/2}$. Integrating, we we want

$$\int_k^1 \; 2\,(1-x^2)^{1/2}\; dx = \pi/4$$

Substitute $x = \sin(\theta)$, and define $\kappa \in (0, \pi/2)$ by $\sin(\kappa) = k$. Of course $\sin(\pi/2) = 1$. So the equation becomes

$$\int_\kappa^{\pi/2} \; 2\,(1-\sin^2(\theta))^{1/2}\;(\cos(\theta)\,d\theta) = \pi/4$$

or

$$\int_\kappa^{\pi/2} \; 2\,\cos^2(\theta)\,d\theta = \pi/4$$

Using the formula in Appendix B, we have

$$2\,\cos^2(\theta) = \cos^2(\theta) + [1 - \sin^2(\theta)] = 1 + [\cos^2(\theta) - \sin^2(\theta)] = 1 + \cos(2\theta)$$

which is just the derivative of $\theta + \frac{1}{2}\,\sin(2\theta)$. Therefore

$$[\pi/2 + \tfrac{1}{2} \times 0] - [\kappa + \tfrac{1}{2}\,\sin(2\kappa)] = \pi/4$$

or

$$\kappa + \tfrac{1}{2}\,\sin(2\kappa) = \pi/4,$$

the same equation as we got in the text using trigonometry.

B: Math reminder – Finding $\cos(2\theta)$ and $\sin(2\theta)$

$$\begin{aligned}
\cos(2\theta) + i\;\sin(2\theta) &= \exp(2\,i\,\theta) \\
&= [\,\exp(i\,\theta)\,]^2 \\
&= [\,\cos(\theta) + i\,\sin(\theta)\,]^2 \\
&= \Big[\;\cos^2(\theta) - \sin^2(\theta)\;\Big] + 2\,i\,\cos(\theta)\,\sin(\theta)
\end{aligned}$$

Therefore

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \quad \text{and} \quad \sin(2\theta) = 2\,\cos(\theta)\,\sin(\theta)$$