# Optimal Policy for Learning by Doing to Reduce Cost of Green Production*

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**Abstract**

This note characterizes the optimal production over time for items of green technologies such as electric cars and solar panels where learning by doing reduces future costs. The optimum requires a subsidy that eventually declines, reaching zero just as learning is completed.

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## The Question

Production of solar panels, electric vehicles, and other similar things with low or no $CO_2$ emission is initially costly, but cumulated experience of production – learning by doing – brings it down. Acquisition of this experience can be speeded up by subsidizing production (and concommitantly consumption) during the early stages. The purpose of this note is to characterize the dynamics of the optimal subsidy policy.

In the context of the whole world facing dangers from climate change, this poses an additional problem. Each country's learning can spill over and benefit other countries' production. In a non-cooperative (Nash) game among countries, this positive externality implies sub-optimal levels of subsidies. We need to know the severity of this problem, and explore ways of overcoming it through international coordination of subsidy policies. This is part of ongoing research.

*I thank Nick Stern for suggesting this problem.

A model of such interaction in learning by doing among profit-maximizing firms was developed by Spence (1981). I will adapt it for my purpose. There are two significant differences. First, I model a government whose objective function is social surplus, whereas Spence considers a profit-maximizing firm. Secondly, while one firm's learning may spill over and benefit others, one firm's production lowers other firms' current profits. So the overall direction of externalities in Spence's model is unclear. There need not be a similar negative externality from one country's learning on other countries, although there could be in a world where nationalism affects international trading relationships. But this is a matter for ongoing research, and not covered in this note.

## One-country (or whole world) model

Let $x$ denote the flow of production/consumption of the item in question, $p$ its price, and $I$ the total income measured in units of a numeraire good. The consumers choose $x$ to maximize a quasilinear utility function

$$U(x) + (I - p\,x) \tag{1}$$

leading to

$$p = U'(x) \tag{2}$$

Let $y$ denote cumulative production up to time $t$:

$$y = \int_0^t x\,dt \tag{3}$$

so

$$\dot{y} = x, \quad y(0) = 0 \tag{4}$$

The cost of production is

$$C(x, y) = L(y)\;x \tag{5}$$

where

$$L(0) = \bar{c}, \quad L'(y) < 0 \text{ for } 0 < y < y^*, \quad L(y) = \underline{c} \text{ for } y \geq y^* \tag{6}$$

Thus learning by doing goes on until a cumulative output $y^*$ is reached, and then ceases. By making total cost proportional to $x$ in (5), I am assuming constant returns to scale in the instantaneous flow of production; this could be generalized but that seems of secondary importance in this context.

The government chooses $x$ (either directly or through tax/subsidy policies) to maximize

$$\int_0^\infty \left[\,U(x) - L(y)\,x\,\right]\; e^{-\delta t}\,dt \tag{7}$$

I solve this by using Pontrjagin's Maximum Principle.[1] The state variable is $y$, the control variable is $x$, and the dynamics of the state variable are given by (4). Denote the

[1]See Dixit (1990) for a simple exposition of the theory, and Shell (1967) for several economic applications.

dual variable by $\psi\, e^{-\delta t}$ (to match the discounting in the objective function). Thus $\psi$ is the shadow value at time $t$ of a marginal unit of $y$ at that time, i.e. the marginal gain that results from increasing $x$ at that time by increasing $y$ and thereby lowering the costs of future production.

Write the Hamiltonian as

$$H = [\ U(x) - L(y)\ x + \psi\ x\ ]\ e^{-\delta t} \tag{8}$$

and choose $x$ to maximize it, which yields

$$U'(x) = L(y) - \psi \tag{9}$$

Think of this as an inverse demand function that equates the marginal utility of consumption at each instant to the "price" on the right hand side. Thus the price at any instant is equal to the current average-cum-marginal cost of production at that instant, minus a subsidy $\psi$ that reflects the shadow value of the contribution that a marginal increase in current production makes to lowering future costs of production by increasing $y$.

The dual variable satisfies the dual differential equation of the Hamiltonian system:

$$d\left(\psi\, e^{-\delta t}\right)/dt = -\partial H/\partial y$$

which simplifies to

$$\dot{\psi} = \delta\ \psi + L'(y)\ x \tag{10}$$

## Solution – an example

I have not so far succeeded in characterizing the solution of this model for a general $L(y)$, but an example brings out useful insights and helps to develop intuition. Assume a linear decline in cost from $\bar{c}$ at $y = 0$ to $\underline{c}$ at $y = y^*$:

$$L(y) = \begin{cases} \bar{c} - k\ y & \text{for } 0 \le y \le y^*, \text{ where } k = (\,\bar{c} - \underline{c}\,)/y^* \\ \underline{c} & \text{for } y^* \le y \end{cases} \tag{11}$$

Then (10) becomes

$$\dot{\psi} = \begin{cases} \delta\,\psi - k\,x & \text{for } 0 \le y \le y^*, \text{ where } k = (\,\bar{c} - \underline{c}\,)/y^* \\ \delta\,\psi & \text{for } y^* \le y \end{cases} \tag{12}$$

It proves useful to work in terms of a new variable

$$z = k\,y + \psi \tag{13}$$

Its differential equation is

$$\dot{z} = k\ \dot{y} + \dot{\psi} = k\,x + \dot{\psi}$$

so

$$\dot{z} = \begin{cases} \delta\,\psi & \text{for } 0 \leq y \leq y^*, \text{ where } k = (\bar{c} - \underline{c})/y^* \\ k\,x + \delta\,\psi & \text{for } y^* \leq y \end{cases} \tag{14}$$

In this example, (9) becomes

$$U'(x) = \begin{cases} \bar{c} - k\,y - \psi = \bar{c} - z & \text{for } 0 \leq y \leq y^* \\ \underline{c} - \psi & \text{for } y^* \leq y \end{cases} \tag{15}$$

Solve for $x$ as a (direct) demand function

$$x = \begin{cases} D(\bar{c} - z) & \text{for } 0 \leq y \leq y^* \\ D(\underline{c} - \psi) & \text{for } y^* \leq y \end{cases} \tag{16}$$

Then (14) becomes

$$\dot{z} = \begin{cases} \delta\,\psi & \text{for } 0 \leq y \leq y^*, \text{ where } k = (\bar{c} - \underline{c})/y^* \\ k\,D(\underline{c} - \psi) + \delta\,\psi & \text{for } y^* \leq y \end{cases} \tag{17}$$

and (12) becomes

$$\dot{\psi} = \begin{cases} \delta\,\psi - k\,D(\bar{c} - z) & \text{for } 0 \leq y \leq y^*, \text{ where } k = (\bar{c} - \underline{c})/y^* \\ \delta\,\psi & \text{for } y^* \leq y \end{cases} \tag{18}$$

Now we can depict possible solution paths for this pair of differential equations in $(z, \psi)$ space. The attached figure shows a general case; other special cases are possible, and one that is significant is noted below.

To understand the figure, note that: [1] $y \geq 0$ corresponds to $z \geq \psi$. So only the region of the figure below the 45-degree line is relevant (although I have shown some curves outside of it for mathematical completeness). The initial point, with $y = 0$, must lie on the 45-degree line. Its actual position (i.e. choice of $\psi(0)$) comes from considerations of optimization. [2] $y = y^*$ corresponds to $z = k\,y^* + \psi$, a line parallel to the 45-degree line and to its right. Below this second line, $y > y^*$ so learning is complete and $L(y)$ has reached its minimum, $\underline{c}$. Between the two parallel lines, learning is ongoing. [3] In the region where learning is going on, the locus of $\dot{\psi} = 0$ is given by $\psi = \frac{k}{\delta}\,D(\bar{c} - z)$, with $\dot{\psi}$ positive above it and negative below it. I have shown this locus in the case where it has a portion lying in the region of relevance (below the 45-degree line), but depending on the values of parameters and the demand function, it may miss this region entirely, in which case $\dot{\psi}$ will be negative throughout the region of relevance. I have also assumed that demand goes to infinity as price goes to zero; therefore the $\psi$-stationary locus asymptotes to the vertical line $z = \bar{c}$, but otherwise it could intersect that line at a finite height. [4] In the region to the right of (or below) the line $z = k\,y^* + \psi$, where learning is complete, the dynamic trajectories of solutions to the system of differential equations (17) and (18) satisfy

$$d\psi/dz = \dot{\psi}\,/\,\dot{z} = (\delta\psi)/[k\,D(\underline{c} - \psi) + \delta\psi]$$

which is positive but less than 1. Therefore these trajectories remain confined to the region: they do not hit the horizontal axis, nor do they cross back into the region where learning is going on. These trajectories are not going to be optimal paths, but it is important to know that they are not going to cause any trouble.[2]

Three examples of possible trajectories of solutions to (17) and (18), with directions of motion indicated by arrows, are shown. The ones shown as dashed thin lines cannot be optimal. The top one continues a subsidy even after learning is complete, and in fact by the logic of the mathematics keeps increasing the subsidy exponentially, resulting in ever more inefficient overproduction/overconsumption as the subsidy is no longer justified by any externality. The bottom path ends the subsidy even though the future benefits from learning are not yet exhausted. The middle one, shown as a thick solid curve, is the correct one. It offers a subsidy while learning is going on: that expands current production, increases cumulative production, and thereby lowers future costs. The subsidy ends exactly when $y$ reaches $y^*$, i.e. $z = k\, y^*$. The dynamics of the subsidy are governed by the variational calculus of optimal control. There can even be an initial phase where the subsidy is increasing (as shown), but for other parameter values the optimal path may lie entirely below the $\psi$-stationary curve , and the subsidy may be decreasing throughout.

## Future work

This is just a starter model; much more needs to be done. (1) More general learning functions $L(y)$ should be explored. (2) The model should be extended to many countries. One country's subsidy will have spillover learning benefits for all, so a non-cooperative Nash (open or closed loop) equilibrium will yield suboptimally small subsidies from a worldwide perspective. (3) That leads us to the question of how cooperation for worldwide optimality can be sustained, especially bearing in mind the heterogeneity of countries – in size, in the levels of technology and already-achieved cost reductions, and so on.

## References

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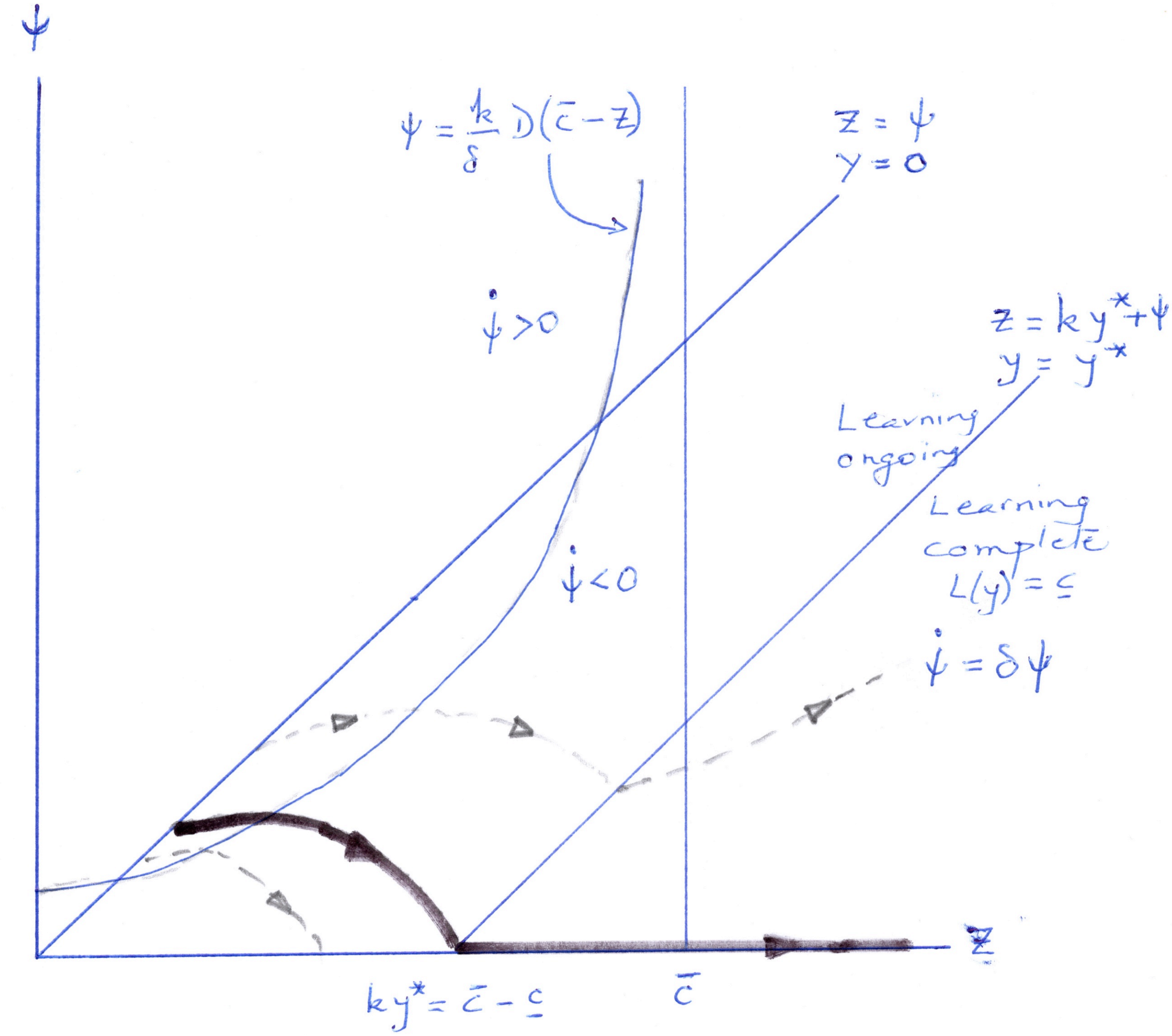
[2]Their curvature depends on the shape of the demand function, but is irrelevant.


$\psi$
$\psi = \frac{k}{\delta} D(\bar{c} - z)$
$z = \psi$
$y = 0$
$\dot{\psi} > 0$
$z = ky^* + \psi$
$y = y^*$
Learning ongoing
Learning complete
$L(y) = \underline{c}$
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$z$
$ky^* = \bar{c} - \underline{c}$
$\bar{c}$