Propagation of uncertainty

David Liao
Harvey Mudd College, Claremont, CA, 91711[*]
(Dated: August 23, 2004)

Review the following as a miniature homework assignment for general laboratory. PU-0.0.1,

I. DERIVATION OF FAMOUS FORMULA

Consider $y = f(x_1, \ldots, x_N)$, a probability density function $\rho(x_1, \ldots, x_N)$ obeying

$$\int dx_1 \cdots dx_N \rho = 1 \tag{1}$$

and definitions from statistics for the average

$$\mu_y \equiv \int dx_1 \cdots dx_N \rho y \tag{2}$$

and the standard deviation

$$\sigma_y^2 \equiv \int dx_1 \cdots dx_N \rho (y - \mu_y)^2 \tag{3}$$

We require that the probability density function is uncorrelated. Mathematically, this implies a factorization

$$\rho(x_1, \ldots, x_N) = \prod_{i=1}^{N} \rho_i(x_i) \tag{4}$$

with corresponding normalization relationships

$$\int dx_i \rho_i(x_i) = 1 \tag{5}$$

averages, and standard deviations. Furthermore, approximate y locally as a toy hyperplane

$$y = \mu_y + \sum \frac{\partial f}{\partial x_i}(x_i - \mu_{x_i}) \tag{6}$$

We want to simplify the expression for σ_y in equation 3. By expanding $(y - \mu_y)^2$, recall that equation 3 can be rewritten as

$$\sigma_y^2 = \int dx_1 \cdots dx_N \rho (y^2 - \mu_y^2) \tag{7}$$

Hence

$$(y^2 - \mu_y^2) = \left[\left(\mu_y + \sum \frac{\partial f}{\partial x_i}(x_i - \mu_{x_i}) \right)^2 - \mu_y^2 \right] \tag{8}$$

Expanding the square

$$\mu_y^2 + 2\mu_y \sum \frac{\partial f}{\partial x_i}(x_i - \mu_{x_i}) + \left[\sum \frac{\partial f}{\partial x_i}(x_i - \mu_{x_i}) \right]^2 - \mu_y^2 \tag{9}$$

reveals a cancellation $\mu_y^2 - \mu_y^2$ and terms $\sum \frac{\partial f}{\partial x_i}(x_i - \mu_{x_i})$ that vanish under integration. Expanding the remaining square reveals auto-terms and cross-terms

$$\left[\sum \frac{\partial f}{\partial x_i}(x_i - \mu_{x_i}) \right]^2 = \left(\frac{\partial f}{\partial x_i}(x_i - \mu_{x_i}) \right)^2 \tag{10}$$

$$+ 2 \frac{\partial f}{\partial x_i}(x_i - \mu_{x_i}) \frac{\partial f}{\partial x_j}(x_j - \mu_{x_j}) + \ldots \tag{11}$$

Using equations 4, 5 we can rewrite the cross-terms as

$$2 \int dx_i \rho_i \frac{\partial f}{\partial x_i}(x_i - \mu_{x_i}) \int dx_j \rho_j \frac{\partial f}{\partial x_j}(x_j - \mu_{x_j}) = 0 \tag{12}$$

Consequently,

$$\sigma_y^2 = \int dx_1 \cdots dx_N \rho \sum \left[\frac{\partial f}{\partial x_i}(x_i - \mu_{x_i}) \right]^2 \tag{13}$$

and using equations 4 and 5,

$$\sigma_y^2 = \sum \int dx_i \rho_i \left[\frac{\partial f}{\partial x_i}(x_i - \mu_{x_i}) \right]^2 \tag{14}$$

we find the famous uncertainty propagation formula

$$\sigma_y^2 = \sum \left[\frac{\partial f}{\partial x_i} \sigma_{x_i} \right]^2 \tag{15}$$

As long as you define $y = f$ and your variables x_i clearly, you need only the assumptions behind equation 15, and equation 15 itself to propagate uncertainty. Use equation 15 recursively (insert "oohs" and "aahs" for humanity's miniscule cleverness). Good riddance to the silliness of pretending that specialized black-box formulas are fundamental.

II. HISTORY

Thanks to Adam Pivonka for identifying dropped summations and unclear explanation.

* URL: http://odin.ac.hmc.edu/~dliao/