Alternating Spatial Patterns for Coordinated Group Motion

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3rd Northeast Control Workshop
University of Pennsylvania GRASP Lab
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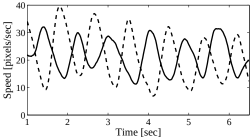
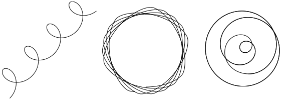
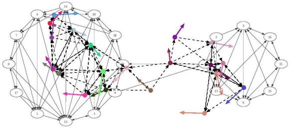
Outline

1. Introduction
   - Data
   - Preview
2. Tools and Models
   - Coupled Oscillator Dynamics
   - Graphs
   - Particles with Coupled Oscillator Dynamics
3. Collective Motion
   - Stability Results
   - Formations
4. Conclusion







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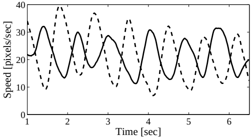
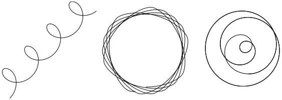
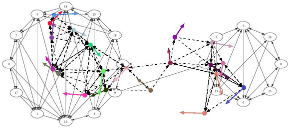
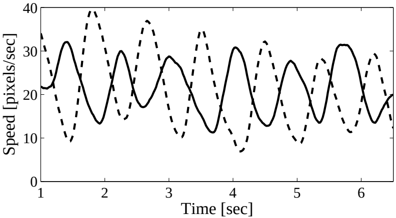




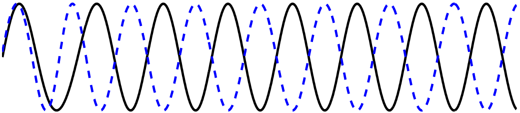


Data



A pair of killifish are observed to actively modulate their speed when swimming together [Couzin, In prep.].

- What kinds of motion does this permit?
- Are there advantages to this?



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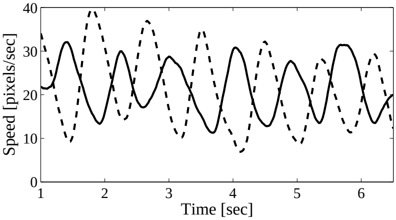

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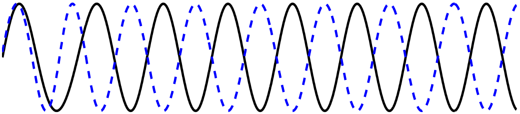

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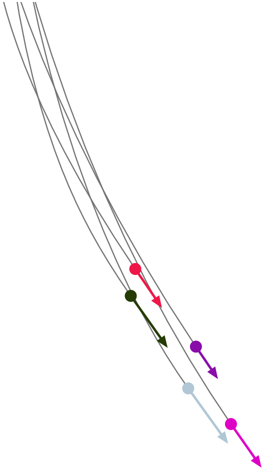
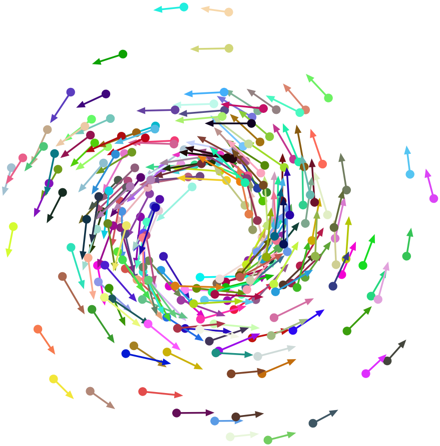
Preview

Particle Model: $\dot{r}_k = (1 + \mu \cos \phi_k)\, e^{i\theta_k}$

$N = 5$



$N = 300$



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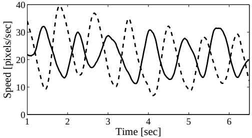

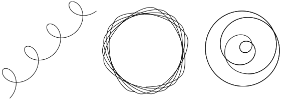

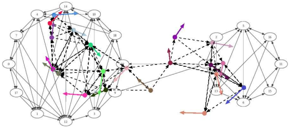

Coupled Oscillator Dynamics

General Context

$\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_N\} \in T^N$

$p_\theta = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}$

$U_1(\boldsymbol{\theta}) = \frac{N}{2} |p_\theta|^2$

$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle = \mathrm{Re}\left\{ \bar{\mathbf{z}}_1^T \mathbf{z}_2 \right\}$

Gradient Control, Natural Frequency ω [Kuramoto, 84]

$$\dot{\theta}_k = \omega - K_\theta \frac{\partial U_1(\boldsymbol{\theta})}{\partial \theta_k} = \omega - K_\theta \left\langle p_\theta, ie^{i\theta_k} \right\rangle = \omega - \frac{K_\theta}{N} \sum_{j=1}^{N} \sin \theta_{jk}$$

$K_\theta < 0, \quad |p_\theta| \to 1$
Synchronized Solution

$K_\theta > 0, \quad |p_\theta| \to 0$
Balanced Solution

Symmetric Patterns
e.g. Splay State [SPL1]

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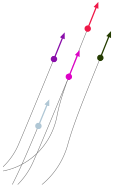

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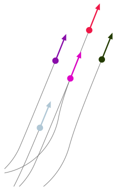

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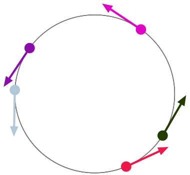
Balanced Solution



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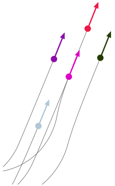
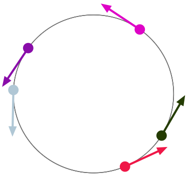
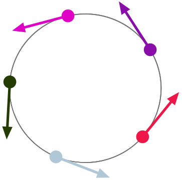
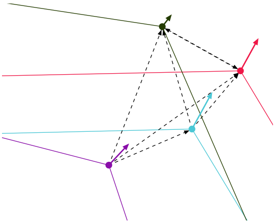
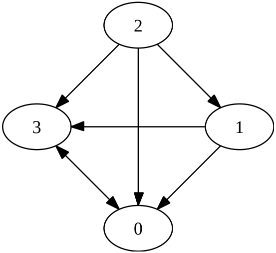
		

Treat the interaction network as a *graph*,
with an edge from i to j if i can *sense* j.



Sample Situation



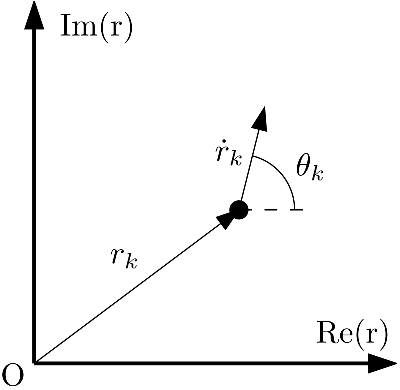
Corresponding Graph

$$L = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & -2 & 0 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Graph Laplacian

Particle Model with Double Oscillators

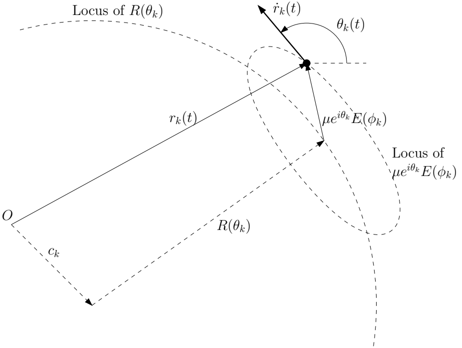
Position: $r_k \in \mathbb{C} \approx \mathbb{R}^2$
Kinematics: $\dot{r}_k = (1 + \mu \cos \phi_k)\, e^{i\theta_k}, \quad 0 \leq \mu < 1$
Control: $\dot{\theta}_k = u_k, \quad \dot{\phi}_k = g_k$

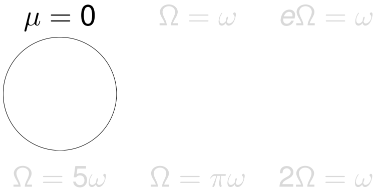


In the absence of external influences, propose that $\dot{\theta}_k = \omega$, $\dot{\phi}_k = \Omega$.

Steady State Orbits

$$r_k = c_k - i\omega^{-1} e^{i\theta_k} + \mu e^{i\theta_k} \frac{\Omega \sin \phi_k + i\omega \cos \phi_k}{\Omega^2 - \omega^2} = c_k + R(\theta_k) + \mu e^{i\theta_k} E(\phi_k)$$





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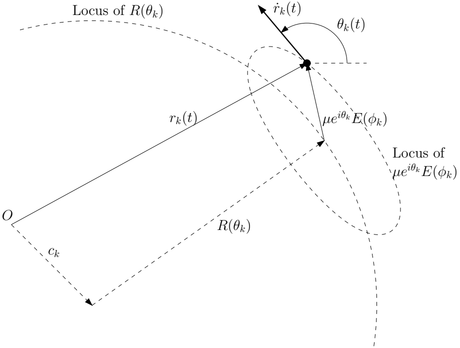

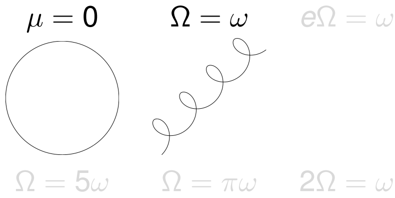

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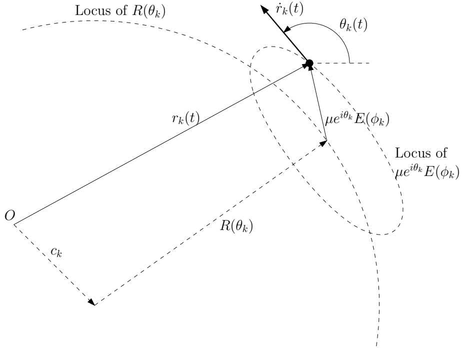

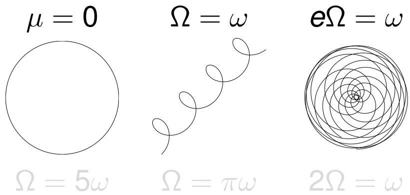

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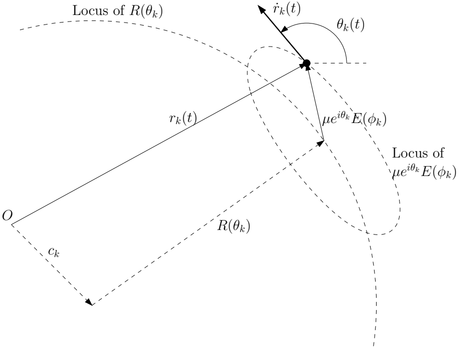

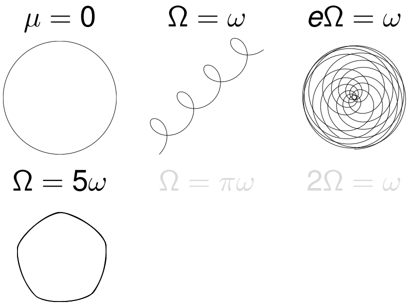

Particles with Coupled Oscillator Dynamics

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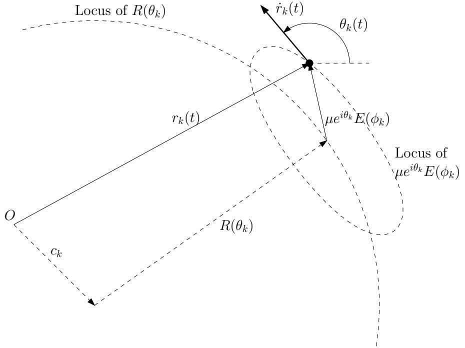

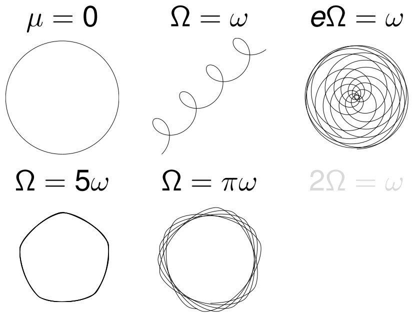

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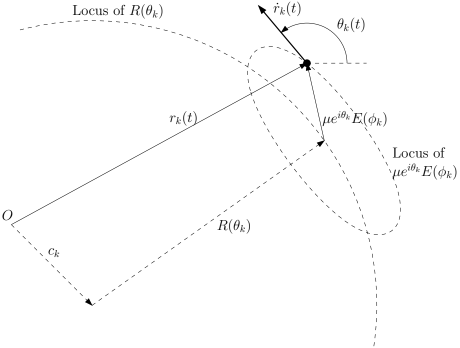

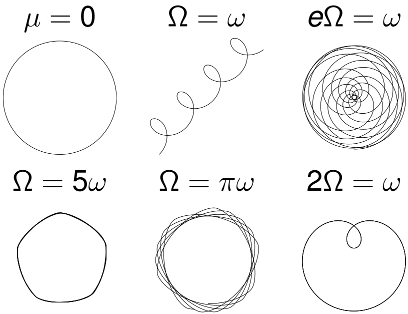

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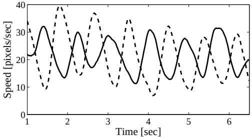

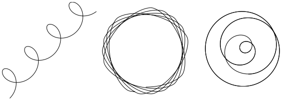

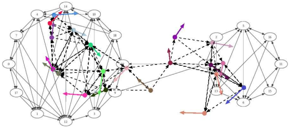

Phase Control

$$\dot{\theta}_k = \omega - K_\theta \frac{\partial U_1(\boldsymbol{\theta})}{\partial \theta_k} = \omega - \frac{K_\theta}{N} \sum_{j=1}^{N} \sin \theta_{jk}$$

$$\dot{\phi}_k = \Omega - K_\phi \frac{\partial U_1(\boldsymbol{\phi})}{\partial \phi_k} = \Omega - \frac{K_\phi}{N} \sum_{j=1}^{N} \sin \phi_{jk}$$

$K_\phi < 0$
ϕ Synchronized

$K_\phi > 0$
ϕ Balanced

$K_\theta < 0$
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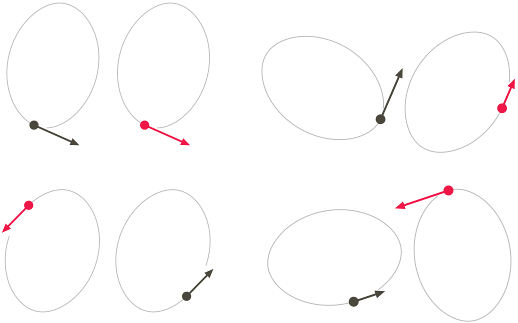

Spacing Control

$$u_k = \omega + \omega\kappa\langle \tilde{c}_k, ie^{i\theta_k}\rangle, \qquad g_k = \frac{\Omega}{\omega} u_k$$

- $\tilde{c}_k = c_k - \frac{1}{N}\sum_{j=1}^{N} c_j$ and $\kappa > 0$
- Recall $r_k = c_k + R(\theta_k) + \mu e^{i\theta_k} E(\phi_k)$
- Lyapunov function $C(\mathbf{r}, \theta, \phi) = \frac{1}{2}\sum_{j=1}^{N} \|\tilde{c}_k\|^2$
- At steady state, $\dot{\theta}_k = \omega$, $\dot{\phi}_k = \Omega$
- No restrictions on θ or ϕ
- Closely related: [SPL1,2]

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$u_k = \omega + \omega\kappa\langle \tilde{c}_k, ie^{i\theta_k}\rangle,$ $\qquad g_k = \frac{\Omega}{\omega} u_k$

- Alternate form:

$$u_k = \kappa \frac{\partial U_1(\theta)}{\partial \theta_k} + \omega \left(1 + \kappa \left\langle \tilde{r}_k, e^{i\theta_k} \right\rangle\right) + \mu\kappa\omega \left\langle \tilde{e}_k, e^{i\theta_k} \right\rangle$$

where $\tilde{r}_k = r_k - \frac{1}{N}\sum_{j=1}^{N} r_j$ and $\tilde{e}_k = e^{i\theta_k} E(\phi_k) - \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} E(\phi_j)$.

- $\frac{\partial U_1(\theta)}{\partial \theta_k}$ depends on relative headings θ_{jk}.
- $\tilde{r}_k$ depends on relative positions r_{jk}.
- $\left\langle \tilde{e}_k, e^{i\theta_k} \right\rangle$ is related to relative speeds and accelerations.

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MAE

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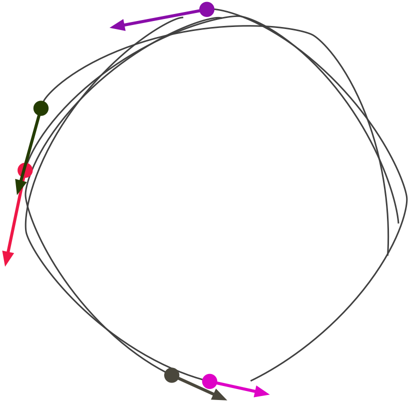
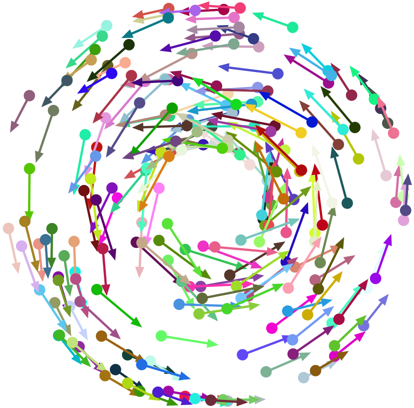
Formations

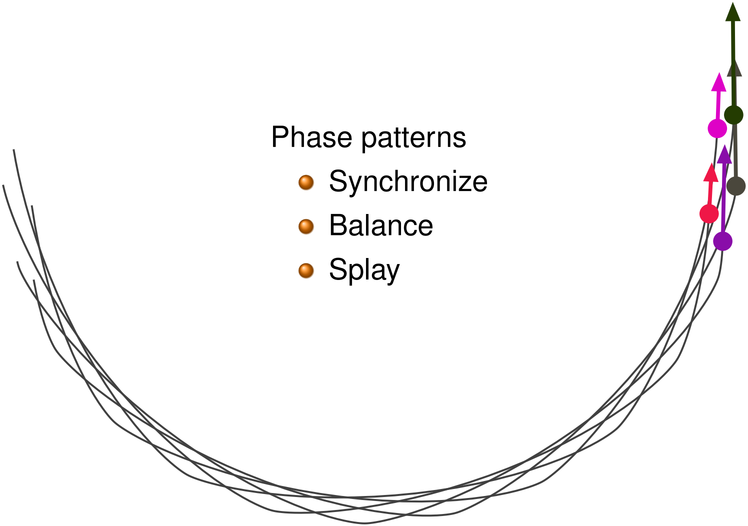
Spacing Control

$N = 5,\ \omega = 1,\ \Omega = 4,\ \mu = 0.9$



$N = 200,\ \omega = 1,\ \Omega = 0.5, \mu = 0.5$

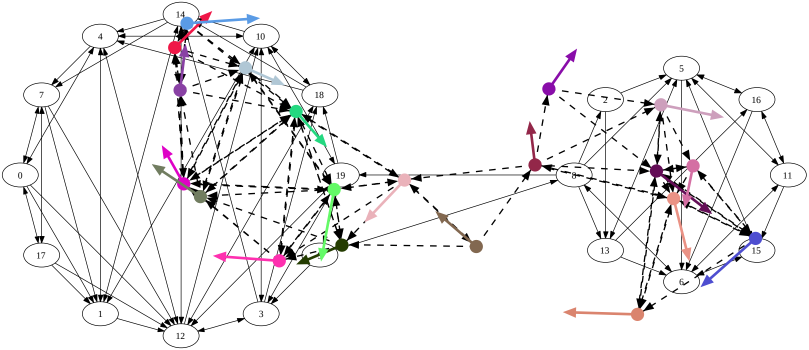




Phase patterns

- Synchronize
- Balance
- Splay

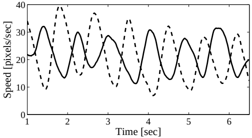
- Extend results to limited communication setting [e.g. SPL2]

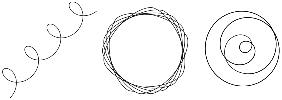


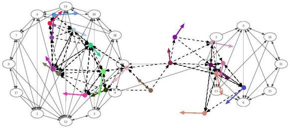
- Alternating patterns may improve information flow
  - Individuals exchange positions over time
  - Dynamics related to graph connectedness

Outline

1. Introduction
   - Data
   - Preview
2. Tools and Models
   - Coupled Oscillator Dynamics
   - Graphs
   - Particles with Coupled Oscillator Dynamics
3. Collective Motion
   - Stability Results
   - Formations
4. Conclusion







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Conclusion

- Speed oscillations are observed in experimental data
- Building from previous results, we can model and coordinate oscillations
- Speed oscillations produce group behavior in which individuals constantly exchange roles
- D.T. Swain, N.E. Leonard, I.D. Couzin, A. Kao, R.J. Sepulchre, “Alternating Spatial Patterns for Coordinated Group Motion”, Submitted, IEEE CDC 2007.