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$$\boldsymbol{\kappa}=\frac{\nabla_{\perp}B}{B}+\frac{4\pi}{c}\frac{\boldsymbol{J}\times\boldsymbol{B}}{B^2} \tag{118}$$

$(118)\cdot\boldsymbol{e}_r$:

$$\kappa_r=\frac{1}{B}\frac{\partial}{\partial r}B+\frac{4\pi}{c}\frac{\sqrt{g}\left(J^\theta B^\zeta-J^\zeta B^\theta\right)}{B^2} \tag{a1}$$

Considering flux representation, $B^\theta=\frac{\chi'}{\sqrt{g}}$, $B^\zeta=\frac{\chi' q}{\sqrt{g}}=qB^\theta$, (a1) becomes

$$\kappa_r=\frac{\partial}{\partial r}lnB-\chi'\frac{4\pi}{cB^2}\left(J^\zeta-qJ^\theta\right) \tag{118-1}$$

using (106), (107) → contravariant component of $\boldsymbol{J}=\frac{c}{4\pi}\nabla\times\boldsymbol{B}$, eq. (118-1) becomes

$$\kappa_r=\frac{\partial}{\partial r}logB-\frac{\chi'}{B^2}\left[\nabla\cdot(R^{-2}\nabla\chi)+\frac{qI'}{\sqrt{g}}\right] \tag{118-2}$$

Considering flux representation again,

$$\boldsymbol{B}=\nabla\zeta\times\nabla\chi+q\nabla\chi\times\nabla\theta \tag{a2}$$

In axisymmetry coordinate(means $\nabla\zeta\cdot\nabla\theta=\nabla\zeta\cdot\nabla r=0$), (a2) can be described,

$$\boldsymbol{B}=\nabla\zeta\times\nabla\chi+I(r)\nabla\zeta \tag{a3}$$

1st term of RHS in (a3) doesn't have B_ζ component(actually we can infer this result easily due to orthogonality - If the coordinate is orthogonal, each covariant and contravariant basis vector points out same direction, but the magnitude of them are different) :

$$\begin{aligned}\nabla\zeta\times\nabla\chi\cdot\boldsymbol{e}_\zeta &=\nabla\zeta\times\nabla\chi\cdot\sqrt{g}(\nabla r\times\nabla\theta)\\ &=\chi'\sqrt{g}(\nabla\zeta\times\nabla r)\cdot(\nabla r\times\nabla\theta)\\ &=\chi'\sqrt{g}\nabla\theta\cdot\{(\nabla\zeta\times\nabla r)\times\nabla r\}\\ &=-\chi'\sqrt{g}\underbrace{\nabla\theta\cdot\nabla\zeta}_{0(\because\text{axisymmetry})}|\nabla r|^2-\nabla r\underbrace{(\nabla\zeta\cdot\nabla r)}_{0(\because\text{axisymmetry})}\}\\ &=0\end{aligned} \tag{a4}$$

Therefore,

$$B_\zeta=I(r) \tag{text 1}$$

In addition,

$$B^\zeta=I(r)|\nabla\zeta|^2=\frac{I(r)}{R^2}=\frac{B_\zeta}{R^2}\quad(|\nabla\zeta|=\frac{1}{R}) \tag{text 2}$$

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Put (text 1) and (text 2) into last term of (118-2).

$$-\frac{\chi'}{B^2}\frac{qI'}{\sqrt{g}}=-\frac{B^\zeta I'}{B^2}=-\frac{B^\zeta B_\zeta'}{B^2} \quad \text{(a5)}$$

If we manipulate $B^\zeta B_\zeta'$ term with $B_T=\frac{B_\zeta}{R}$,

$$\begin{aligned} B^\zeta B_\zeta' &= B^\zeta \frac{\partial}{\partial r}B_\zeta = \frac{B_\zeta}{R^2}\frac{\partial}{\partial r}B_\zeta = \frac{1}{2}\frac{1}{R^2}\frac{\partial}{\partial r}B_\zeta^2 \\ &= \frac{1}{2}\left\{\frac{\partial}{\partial r}\frac{B_\zeta^2}{R^2}-B_\zeta^2\frac{\partial}{\partial r}\frac{1}{R^2}\right\}=\frac{1}{2}\left\{\frac{\partial}{\partial r}B_T^2+2\frac{B_\zeta^2}{R^3}\frac{\partial}{\partial r}R\right\} \\ &= \frac{1}{2}\frac{\partial}{\partial r}B_T^2+\frac{B_T^2}{R}\frac{\partial}{\partial r}R=\frac{1}{2}\frac{\partial}{\partial r}B_T^2+B_T^2\frac{\partial}{\partial r}lnR \end{aligned} \quad \text{(118-3)}$$

and we could manipulate 1st term of RHS in (118-2) like

$$\begin{aligned} \frac{\partial}{\partial r}lnB &= \frac{\partial B^2}{\partial r}\frac{\partial}{\partial B^2}lnB \\ &= \frac{\partial B^2}{\partial r}\frac{\partial}{\partial B^2}ln\sqrt{B^2}=\frac{\partial B^2}{\partial r}\frac{1}{2B^2}=\frac{1}{2B^2}\frac{\partial}{\partial r}(B_p^2+B_T^2) \end{aligned} \quad \text{(118-4)}$$

using the results (118-3) and (118-4), we can obtain

$$\kappa_r=\frac{1}{2B^2}\frac{\partial B_p^2}{\partial r}-\frac{B_T^2}{B^2}\frac{\partial}{\partial r}lnR-\frac{\chi'}{B^2}\nabla\cdot(R^{-2}\nabla\chi) \quad \text{(118-5)}$$

From now on to the (119), the manipulation is straight forward.