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By definition, $(\boldsymbol{b}, \hat{\boldsymbol{s}}, \hat{\boldsymbol{\rho}})$ is orthogonal system.
From eq. (49),

$$\frac{d}{dt}(\boldsymbol{u}+\boldsymbol{s}) = \frac{e}{m}\boldsymbol{E} - \Omega s\hat{\boldsymbol{\rho}} \tag{49}$$

$\boldsymbol{s}\cdot(49)$:

$$\boldsymbol{s}\cdot\frac{d}{dt}(\boldsymbol{u}+\boldsymbol{s}) = \frac{e}{m}\boldsymbol{s}\cdot\boldsymbol{E} - \Omega s\boldsymbol{s}\cdot\hat{\boldsymbol{\rho}} \tag{a1}$$

0(∵orthogonal)

$$\rightarrow u\boldsymbol{s}\cdot\frac{d\boldsymbol{b}}{dt} + \frac{d}{dt}\frac{1}{2}s^2 = \frac{e}{m}\boldsymbol{s}\cdot\boldsymbol{E} \tag{50}$$

$$\rightarrow \frac{1}{B}\frac{d}{dt}\frac{1}{2}ms^2 = -\frac{\mu}{B}\boldsymbol{s}\cdot\frac{d\boldsymbol{b}}{dt} + \frac{e}{B}\boldsymbol{s}\cdot\boldsymbol{E}$$

$$\rightarrow \frac{d\mu}{dt} = -\frac{\mu}{B}\frac{dB}{dt} - \frac{\mu}{B}\boldsymbol{s}\cdot\frac{d\boldsymbol{b}}{dt} + \frac{e}{B}\boldsymbol{s}\cdot\boldsymbol{E}$$

$\boldsymbol{v}\cdot(49)$:

$$\boldsymbol{v}\cdot\frac{d}{dt}(\boldsymbol{u}+\boldsymbol{s}) = \frac{e}{m}\boldsymbol{v}\cdot\boldsymbol{E} - \Omega s\boldsymbol{v}\cdot\hat{\boldsymbol{\rho}} \tag{a2}$$

0(∵orthogonal)

$$\rightarrow \frac{d}{dt}\frac{1}{2}mv^2 = e\boldsymbol{v}\cdot\boldsymbol{E} \tag{51}$$

$$\rightarrow \frac{dU}{dt} = e\frac{d\Phi}{dt} + e\boldsymbol{v}\cdot\left(-\nabla\Phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right)$$

$$\rightarrow \frac{dU}{dt} = e\frac{\partial\Phi}{\partial t} + e\boldsymbol{v}\cdot\nabla\Phi + e\boldsymbol{v}\cdot\left(-\nabla\Phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right)$$

※errata in the book

$$\rightarrow \frac{dU}{dt} = e\frac{\partial\Phi}{\partial t} - \frac{e}{c}\boldsymbol{v}\cdot\frac{\partial\boldsymbol{A}}{\partial t}$$

$\hat{\boldsymbol{\rho}}\cdot(49)$:

$$\hat{\boldsymbol{\rho}}\cdot\frac{d}{dt}(\boldsymbol{u}+\boldsymbol{s}) = \frac{e}{m}\hat{\boldsymbol{\rho}}\cdot\boldsymbol{E} - \Omega s\hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{\rho}} \tag{a3}$$

We need to be careful to manipulate LHS; RHS is just as it is. Considering orthogonal system, it can be described as

$$u\hat{\boldsymbol{\rho}}\cdot\frac{d\boldsymbol{b}}{dt} + s\hat{\boldsymbol{\rho}}\cdot\frac{d\hat{\boldsymbol{s}}}{dt} \tag{a4}$$

By definition from eq. (47), the second term of (a4) can be expressed by

$$\begin{aligned} s\hat{\boldsymbol{\rho}}\cdot\frac{d\hat{\boldsymbol{s}}}{dt} &= s\hat{\boldsymbol{\rho}}\cdot\frac{d}{dt}(\boldsymbol{e}_2 cos\gamma - \boldsymbol{e}_3 sin\gamma) \\ &= s\frac{d\gamma}{dt}\hat{\boldsymbol{\rho}}\cdot(-\boldsymbol{e}_2 sin\gamma - \boldsymbol{e}_3 cos\gamma) + \left(s\cos\gamma\hat{\boldsymbol{\rho}}\cdot\frac{d\boldsymbol{e}_2}{dt} - s\sin\gamma\hat{\boldsymbol{\rho}}\cdot\frac{d\boldsymbol{e}_3}{dt}\right) \\ &= -s\frac{d\gamma}{dt} + s\left(\cos^2\gamma\boldsymbol{e}_3\cdot\frac{d\boldsymbol{e}_2}{dt} - \sin^2\gamma\boldsymbol{e}_2\cdot\frac{d\boldsymbol{e}_3}{dt}\right) \end{aligned} \tag{a5}$$

For next step and the derivation of last line of (a5), we need to know following vector identities :

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$$\boldsymbol{e}_i \cdot \frac{d\boldsymbol{e}_i}{dt} = \frac{1}{2}\frac{d}{dt}(\boldsymbol{e}_i \cdot \boldsymbol{e}_i) \underset{\uparrow}{=} \frac{d}{dt}1 = 0 \quad \text{(a6)}$$

orthogonal system, unit vector

$$\frac{d}{dt}(\boldsymbol{e}_2 \cdot \boldsymbol{e}_3) = 0$$

$$\rightarrow \frac{d\boldsymbol{e}_2}{dt} \cdot \boldsymbol{e}_3 + \boldsymbol{e}_2 \cdot \frac{d\boldsymbol{e}_3}{dt} = 0 \quad \text{(a7)}$$

$$\rightarrow \frac{d\boldsymbol{e}_2}{dt} \cdot \boldsymbol{e}_3 = -\boldsymbol{e}_2 \cdot \frac{d\boldsymbol{e}_3}{dt}$$

Therefore, with (a7), (a5) can be

$$s\hat{\boldsymbol{\rho}} \cdot \frac{d\hat{\boldsymbol{s}}}{dt} = -s\frac{d\gamma}{dt} + s\boldsymbol{e}_3 \cdot \frac{d\boldsymbol{e}_2}{dt} \quad \text{(a8)}$$

Finally, (a3) with (a4) and (a8) becomes,

$$\frac{d\gamma}{dt} = \Omega + \boldsymbol{e}_3 \cdot \frac{d\boldsymbol{e}_2}{dt} - \frac{u}{s}\hat{\boldsymbol{\rho}} \cdot \frac{d\boldsymbol{b}}{dt} - \frac{e}{ms}\hat{\boldsymbol{\rho}} \cdot \boldsymbol{E} \quad \text{(52)}$$