Lecture 6: Urban Structure and Growth

Economics 522

Esteban Rossi-Hansberg

Princeton University

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Lucas and Rossi-Hansberg (2002)

- Analyzes the economic forces that determine the internal structure of cities
- Circular city where a single good is produced using land and labor
- Basic elements:
  - People consume goods and residential land
  - There is a production externality
  - Workers face commuting costs
The total land area of the city, $\pi S^2$, is divided between production use and residential use.

We describe locations within the city by their polar coordinates, $(r, \phi)$.

For any location $r$, let

- $\theta(r)$ be the fraction of land used for production.
- $n(r)$ be the employment density—employment per unit of production land—at location $r$.
- $N(r)$ be the number of workers housed at $r$, per unit of residential land.
- $\ell(r)$ units of land per person.
- $c(r)$ consumption per person.
City Structure
Agglomeration

- The production externality is assumed to be linear, and to decay exponentially at a rate $\delta$ with the distance between $(r,0)$ and $(s,\phi)$:

$$z(r) = \delta \int_0^S \int_0^{2\pi} s\theta(s, \phi) n(s, \phi) e^{-\delta x(r, s, \phi)} d\phi ds,$$

where

$$x(r, s, \phi) = \left[ r^2 - 2\cos(\phi)rs + s^2 \right]^{1/2}.$$

Since allocations are symmetric, we can write

$$z(r) = \int_0^S \varphi(r, s) s\theta(s) n(s) ds,$$

where

$$\varphi(r, s) = \delta \int_0^{2\pi} e^{-\delta x(r, s, \phi)} d\phi.$$
Commuting Costs

- A person employed at $r$ and living at $s$ must travel the distance $|r - s|$ twice daily. Assume that such a worker supplies

$$e^{-\kappa |r - s|} \approx 1 - \kappa |r - s|$$

hours of labor at location $r$

- $H(r)$ is the stock of workers that remain unhoused at $r$, after employment and housing have been determined for locations $s \in [0, r)$

- Let

$$y(r) = 2\pi r [\theta(r)n(r) - (1 - \theta(r))N(r)],$$

then

$$\frac{dH(r)}{dr} = y(r) + \kappa H(r) \quad \text{if} \quad H(r) > 0,$$

$$\frac{dH(r)}{dr} = y(r) - \kappa H(r) \quad \text{if} \quad H(r) < 0,$$

$H(0) = 0$ and $H(S) \leq 0$. 
Let

\[ S^+ = \text{cl}\{r \in [0, S] : H(r) > 0\} \]

(where \(\text{cl}\{A\}\) denotes the closure of \(A\)),

\[ S^- = \text{cl}\{r \in [0, S] : H(r) < 0\} \]

and

\[ S^0 = \{r \in [0, S] : H(r) = 0 \ \text{and} \ y(r) = 0\} \]
Firm Problem and Household Problem

- Firm Problem at location $r$

\[
q(r) = g(z(r))f(n(r)) - w(r)n(r)
\]

\[
= \max_n \{g(z(r))f(n) - w(r)n\}
\]

where $w(r)$ is the wage rate and $q(r)$ is the business bid rent. Let \( \hat{n}(w(r), z(r)) \) and \( \hat{q}(w(r), z(r)) \) be the maximizing values.

- Household’s problem at $r$

\[
w(r) = c(r) + Q(r)\ell(r) = \min_{c, \ell} [c + Q(r)\ell],
\]

subject to

\[
U(c, \ell) \geq \bar{u},
\]

where $Q(r)$ is the residential bid rent. Let \( \hat{N}(w(r)) \) and \( \hat{Q}(w(r)) \) be the maximizing values.
Land use and Wages

- **Land use**: It will be assumed that land is allocated to its highest-value use. In context, this means that

\[ \theta(r) > 0 \quad \text{implies} \quad q(r) \geq Q(r), \]

and

\[ \theta(r) < 1 \quad \text{implies} \quad q(r) \leq Q(r). \]

- **Wage no arbitrage condition**: Free mobility of labor implies a sharp restriction on equilibrium wages \( w(r) \):

\[ e^{-\kappa |r-s|} w(s) \leq w(r) \leq e^{\kappa |r-s|} w(s) \]

for all \( r, s \in [0, S] \).
An **equilibrium** is a pair of piecewise continuous functions $\theta$ and $y$, and a collection $(z, n, N, w, q, Q, H)$ of continuous functions, all on $[0, S]$ such that for all $r$,

1. $w(r)$ satisfies wage no arbitrage condition,
2. $n(r) = \hat{n}(w(r), z(r))$ and $q(r) = \hat{q}(w(r), z(r))$,
3. $N(r) = \hat{N}(w(r))$ and $Q(r) = \hat{Q}(w(r))$,
4. $\theta(r), q(r)$, and $Q(r)$ satisfy $0 \leq \theta(r) \leq 1$, land use conditions
5. $y(r), n(r), N(r), \theta(r)$ and $H(r)$ are constructed as above
6. $H(S) = 0$, and
7. $z, \theta$, and $n$ satisfy the production externality equation
Assumptions

The functions \( \hat{n}, \hat{q} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \) and \( \hat{N}, \hat{Q} : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \), are continuously differentiable. Both \( \hat{n} \) and \( \hat{q} \) are decreasing in \( w \) and increasing in \( z \); both \( \hat{N} \) and \( \hat{Q} \) are increasing in \( w \).

The first derivative \( \hat{n}_z(w, z) \) satisfies

\[
\lim_{z \to \infty} \hat{n}_z(w, z) = 0,
\]

and

\[
\lim_{z \to 0} \hat{n}_z(w, z) = \infty,
\]

for all \( w > 0 \).

The function \( U \) is homogeneous of degree one, and \( U(c, 1) \) is strictly increasing in \( c \).
Existence of Equilibrium

Theorem

There exist a unique allocation that satisfies (1)-(4)

- The wage path

\[
\begin{align*}
    w(r) &= Ke^{-kr} \text{ if } r \in S^+, \\
    w(r) &= Ke^{kr} \text{ if } r \in S^-, \\
    w_m(r) &= \hat{w}_m(z(r)) \text{ if } r \in S^0.
\end{align*}
\]

where \( \hat{w}_m(z(r)) \) solves

\[
\hat{q}(w(r), z(r)) = \hat{Q}(w(r))
\]
Wage Paths

FIGURE 2: EQUILIBRIUM WAGE DETERMINATION

Along the curve Wm(r), land is equally valuable in business and residential use. Above the curve Wm(r), land is more valuable in residential use; below the curve, it is more valuable in business use.

On any interval, an equilibrium wage path must (a) increase at the rate kappa (unless, it meets the path Wm(r)) (b) decrease at the rate kappa (unless, it meets the path Wm(r)), or (c) coincide with the path Wm(r).

Along a path increasing at rate kappa, people travel from left to right to get to work. Along a path decreasing at kappa, people travel right to left to get to work.
A shooting algorithm for constructing an equilibrium wage path:
(1) Pick a wage at r = 0: \( w_0 \) (say).
(2) Continue to the right in the only possible way;
(3) Keep track of the implied path \( H(r, w_0) \) of the stock of unhoused workers.
(4) When the path crosses \( W_m(r) \) into a residential area just keep going.
(5) When \( H(r, w_0) = 0 \), rate of change of wages changes sign
(6) Call stock \( H(S, w_0) = \text{phi}(w_0) \).
Wage Paths

FIGURE 5a: EQUILIBRIUM WAGE DETERMINATION

An example of a family of wage paths consistent with mixed use and branching.

Wm(r)

Distance from city center, miles

0 1 2 3 4 5 6 7 8 9 10

1 1.2 1.4 1.6 1.8 2 2.2 2.4 2.6 2.8 3

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Equilibrium Given Productivity

- Unique wage path satisfies

\[ H(S) = 0 \]

**FIGURE 6**

The correspondence \( \varphi(\omega) \) from the initial wage rate \( \omega \) at \( r = 0 \) to the terminal stock \( H(S,\omega) \) of unhoused workers at \( r = S \).
Example

FIGURE 7: LINEAR PRODUCTIVITY EXAMPLE

Kappa = .005

H(r) and y(r)

Distance from city center, miles

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Equilibrium

- An equilibrium has to satisfy
  \[ (Tz)(r) = \int_0^S \varphi(r, s)s\theta(s; z)n(s; z)ds \]

- \( T \) maps nonnegative valued functions into nonnegative valued functions
- \( T \) is continuous in the sup norm as \( w(r; z) \) is continuous in \( z \)
- \( w(r; z) \) is increasing in \( z \) (point by point)
- If \( z(r) < \bar{z} \) all \( r \) then \( (Tz)(r) < \bar{z} \)
- Then by Shauder’s fixed point theorem equilibrium there exist a function \( z^* \) such that \( z^* = Tz^* \)

Theorem

An equilibrium exists
Equilibrium

**FIGURE 9: ITERATES OF THE OPERATOR T**

- Kappa = 0.005
- Gamma = 0.04
- Delta = 5

- Initial Productivity Function z(r)
- Fixed Point
- (Tz)(r)
Numerical Examples

- Functional forms:

\[ U(c, \ell) = c^\beta \ell^{1-\beta} \]
\[ g(z) = z^\gamma \]
\[ f(n) = n^\alpha \]

- Calibration: \( \alpha = .95, \beta = .9, \gamma = .04, A = \bar{u} = 1, S = 10. \) In order for assumption A to hold we need

\[ 0 < \gamma < 1 - \alpha. \]

- Results are very sensitive to \( \kappa \)
- Increasing \( \delta \) concentrates production areas
Numerical Examples

Figure 10: Land Rents, Various Delta Values, Kappa = .001

- Delta = 15
- Delta = 5
- Delta = 10

Distance from city center, miles

Gamma = 0.04
Numerical Examples

**FIGURE 11: LAND RENTS, VARIOUS DELTA VALUES, KAPPA = .005**

- Gamma = 0.04
- Delta = 5
- Delta = 10
- Delta = 15

Land Rents vs. Distance from city center, miles
Numerical Examples

**Figure 13:** Land Rents, Various Delta Values, Kappa = 0.005

- Gamma = 0.04
- Delta = 25
- Delta = 30
- Delta = 40

Distance from city center, miles

Land Rents
Eeckhout (2004)
Gibrat’s Law
Another Look
Another Look
Right Tail Close to Pareto
A Simple Theory

- Let there be a set of locations (cities) $i \in I = \{1, ..., I\}$
- Each city has a continuum population of size $S_{i,t}$
- Total country-wide population is $S = \sum I S_{i,t}$
- All individuals are infinitely lived and can perform exactly one job
- $A_{i,t}$ is the productivity of city $i$ at time $t$ with

$$A_{i,t} = A_{i,t-1} (1 + \sigma_{i,t})$$

where $\sigma_{i,t}$ is an exogenous productivity shock
- Denote by $\sigma_t$ the vector of shock by all cities
- Shock is symmetric, iid, mean zero and $1 + \sigma_{i,t} > 0$
- No aggregate growth in productivity
A Simple Theory

The marginal product of a worker is given by

$$y_{i,t} = A_{i,t}a_+ (S_{i,t})$$

- $a_+ (S_{i,t}) > 0$ is the positive external effect
- Denote the wage by $w_{i,t}$, then $w_{i,t} = y_{i,t}$ as firms are competitive
- Large cities have higher wages
- Workers have one unit of time and work $l_{i,t} \in [0, 1]$
- Some work is lost because of commuting, so productive labor is

$$L_{i,t} = a_- (S_{i,t}) l_{i,t}$$

where $a_- (S_{i,t}) \in [0, 1]$ and $a'_- (S_{i,t}) < 0$ is a negative external effect
Consumer Maximization

- Land in a city is fixed at $H$
- Price of land given by $p_{i,t}$ and an individual's consumption of land by $h_{i,t}$
- Consumers and firms are perfectly mobile
- Consumer solve

$$\max u (c_{i,t}, h_{i,t}, l_{i,t}; S_i) = c_{i,t}^{\alpha}, h_{i,t}^{\beta} (1 - l_{i,t})^{1-\alpha-\beta}$$
$$s.t. c_{i,t} + p_{i,t} h_{i,t} \leq w_{i,t} L_{i,t}$$

- Perfect mobility implies that

$$u^* (S_{i,t}) = U \text{ all } i, t$$

and so

$$A_{i,t} a_+ (S_{i,t}) a_- (S_{i,t}) S_{i,t}^{-\frac{\beta}{\alpha}} \equiv A_{i,t} \Lambda (S_{i,t})$$

is constant across cities
City Size

- This implies that

\[ S_{i,t} \Lambda^{-1}(A_{i,t}) = K \]

\[ S_{i,t} \Lambda^{-1}(A_{i,t-1}(1+\sigma_{i,t})) = K \]

- So \( \Lambda' < 0 \) implies that

\[ \frac{dS_{i,t}}{d\sigma_{i,t}} > 0 \]

- If \( \Lambda \) is a power function

\[ \Lambda^{-1}(A_{i,t}) = \Lambda^{-1}(A_{i,t-1}) \Lambda^{-1}(1+\sigma_{i,t}) \]

- So

\[ S_{i,t} = \frac{K}{\Lambda^{-1}(A_{i,t-1}) \Lambda^{-1}(1+\sigma_{i,t})} \]

\[ = \frac{K}{\Lambda^{-1}(1+\sigma_{i,t})} S_{i,t-1} \]

\[ \equiv (1 + \varepsilon_{i,t}) S_{i,t-1} \]
Gibrat’s Law and the Size Distribution

- Taking natural logarithms and letting, \( \ln (1 + \varepsilon_{i,t}) \approx \varepsilon_{i,t} \) for \( \varepsilon_{i,t} \) small,

\[ \ln S_{i,t} \approx \ln S_{i,t-1} + \varepsilon_{i,t} \]

and so

\[ \ln S_{i,T} \approx \ln S_{i,0} + \sum_{t=1}^{T} \varepsilon_{i,t} \]

- But then, since shocks are iid the Central Limit Theorem implies that

\[ \ln S_{i,T} \sim N \]