Issued Under Office of Naval Research
Contract No. Nonr 1858(16)

A PRICE LEADERSHIP METHOD FOR SOLVING THE INSPECTOR'S NON-CONSTANT-SUM GAME

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Econometric Research Program
Research Memorandum No. 60
September 5, 1963

Research performed in part for Mathematica, and was supported partially by the Office of Naval Research and partially by The Carnegie Corporation of New York.

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Abstract

An inspector's game is a non-constant sum 2-person game in which one player has promised to perform a certain duty and the other player is allowed to occasionally inspect and verify that the duty has indeed been performed.

A solution to a variant of such a game is given in this paper, based on the assumption that the inspector can announce his mixed strategy in advance, if he so wishes, whereas the other player, who has already given his promise, cannot threat by explicitly saying that he will not keep his word.

1. Introduction.

The theory of constant-sum 2-person games exhibits a large number of examples which can be called "hide and seek games". In such games, one player is trying to hide an object or camouflage an action, while the other player tries to discover the hidden object or to uncover the action.

In many real-life cases, however, there is an additional aspect to the situation, which should not be ignored; namely, it is conceivable that the object is not hidden at all or that the action is not taking place. This possibility accounts for the fact that such a game is, in general, a non-constant-sum game, for which today there does not exist a theory as solid as for the constant-sum game theory.

It is precisely for the fact that most real-life situations are not constant-sum games that Game Theory is sometimes being criticized - unjustifiably in my opinion - of being of very little use in analyzing people's conflicts.

The purpose of this paper is to describe a non-constant-sum game, of the "hide and search" type in which it is possible to recommend a strategy which is, in our opinion, as convincing as the minimax strategy for the constant-sum games.

Take for example a treaty between two countries, in which some kind of disarmament is agreed upon, and each side is allowed to inspect the territory of the other party on certain occasions, in order to give some guarantee that no violation occurs.

From the point of view of Game Theory, such a treaty exhibits the following interesting features:

(i) The main purpose of the treaty is that at least "the other country" will not violate its promise. It is also hoped that, if obeyed, this treaty will lead to other agreements towards more disarmament, towards controlling the arms race, and, in general, towards achieving better prospects for peaceful coexistence. It is therefore safe to say that the utility of an obeyed treaty is higher than the utility gained if the "other side" is violating it, <u>even if it is caught</u>.

(ii) There is a clear asymmetry in the possibility of communication between the inspection role and the (possible) violation role. The inspector can always announce in advance, if he so wishes, and also commit himself to his mixed strategy choice of using his available inspections, whereas the violator who wishes to violate secretly, cannot announce his mixed strategy, i. e., his probability distribution on conducting secret violations under the various possibilities. After all, he has just signed a treaty which he solemnly promised to obey.

It turns out that this game theory formulation and the hinted suggestion that "truth is the best lie", is the key to a relatively easy and convincing method to arrive at a "best" strategy for the inspector.

Our game model is defined in Section 2, and it is described for a somewhat general class of real life situations (to which it can be applied perhaps even better than to the problems involved in the above treaty). Its solution is discussed in Sections 3 and 4, and its payoffs are computed and their properties are discussed in Sections 5 and 6. Sections 7, 8, 9 deal with difficulties which may arise in real-life situations.

Our logical steps can be summarized as follows:

(i) We advocate that the inspector announce his choice of mixed strategy in a binding way, prior to playing the game.

(ii) This leaves the suspected violator in a 1-person game which is easily solved as a maximum problem. Thus, for each strategy choice of the inspector, it is possible to compute the expected payoff to both parties.

(iii) Naturally, the inspector should announce that mixed strategy which will yield him the highest expected payoff. Such a strategy is easy to compute.

This procedure is an analogue to the price leadership procedure in the duopoly problems in Economics.

(iv) It turns out that in our particular model, the inspector has to announce "almost" that strategy which is a minimax strategy based on the suspected violator's payoff matrix, regarded as a constant-sum game. Thus he should announce a strategy which is in essence opposing the other party's interests. The "almost" above comes to indicate that his announced strategy is slightly modified in order to force the suspected violator to use the wait-until-all-the-available-inspections-were-used-up strategy.

(v) It turns out that the expected payoffs are Pareto optimal; namely, any other outcome is less preferred by at least one of the players to the outcome which we advocate.

These facts furnish an answer to the suggestion that the inspector might have done even better by not disclosing his mixed strategy in advance. Indeed, he could have obtained essentially more than what our procedure guarantees, only if the suspected violator were to receive less than the amount that our method provides him. But certainly, the suspected violator can guarantee himself this amount by playing a minimax strategy based on his own payoff matrix.

Thus, announcing his mixed strategy will not harm the inspector (essentially). Moreover, by not announcing his strategy, the inspector may actually obtain a smaller payoff (Section 7).

I wish to express my indebtedness to Professor H. W. Kuhn and to Dr. M. Davis for various discussions that stimulated this paper.

2. The Inspector's Game.

This is a 2-person non-constant-sum game, whose players are called the inspector and the violator. These players are assumed to have signed a treaty in which the violator promises to obey certain duties (for which he gets compensation from the inspector). In order to give some assurance that the treaty is not secretly violated, the inspector is allowed to conduct a certain amount of "on site" inspections on suspicious occasions.

We make the following simplifying assumptions:

Assumption 1. There are exactly n "suspicious events", which are known to both sides, and, as far as the inspector is concerned could equally likely indicate a possible violation. No violation can otherwise occur.

Assumption 2. The violator can make at most one of these events, and no others, as a secret violation, and this violation will not cause this event to look more suspicious to the inspector.

Assumption 3. The treaty specifies that the inspector is permitted r inspections on these events, $0 \leq r \leq n$. If a secret violation has occurred and has been inspected, it will be identified with a certain fixed positive probability.

The following assumptions are made on the utility payoffs.

Assumption 4. (i) If the violator violates [1] on event i, and the inspector inspects this event, then the payoff to the violator is 0 units of utility and the payoff to the inspector is 1 unit of utility. (ii) If the violator violates on event i, and the inspector does not inspect this

(1) "Violates" will mean henceforth conducting a secret act which is contrary to the terms of the treaty.

event, then the payoff to the violator is 1 unit of utility and the payoff to the inspector is 0 units of utility.

Since the zeroes and the units of utility can be chosen at will for each player, this assumption states then that if an event i is a violation, then the violator prefers that i is not inspected whereas the inspector prefers that it is inspected. It also states that "other things being the same", it is irrelevant to both players which of the events is a violation.

Assumption 5. Whether an inspection has occurred on event i or not is known to the violator after event i and prior to event i+1. The violator can make his decision at any moment and it is not known to the inspector, unless a violation is inspected and is identified. An inspection of an unviolated event, by itself[1], has no effect on the payoffs to both players.

By the last statement we ignore many "psychological" aspects of inspections which may exist in real-life situations, such as: losing

(1) I. e., aside from the fact that it reduces the number of the available inspections and that it yields the information that it existed.

face caused by inspecting and not discovering a violation, humiliating the violator by treating him as a suspected liar, building up a trust or distrust by inspecting "rarely" or "too often", etc. It is also assumed that the inspection can serve no purposes[(1)] other than discovering a violation.

As is usually (but not always) the case, we assume that the primary purpose of the treaty is that it be observed. The inspections play a secondary protective role. Accordingly, we denote by α and β, respectively, the payoffs in utility units to the inspector and to the violator, if no violation took place during the n events. We make the following

Assumption 6. The utilities α and β are known to both parties[(2)] and they satisfy

$$\alpha > 1 \ , \quad 0 < \beta < 1 \ . \tag{2.1}$$

The assumption $\alpha > 1$ means that the inspector would prefer that the treaty is observed to a situation in which a violation occurs and it is inspected.

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(1) Such as spying, sabotaging the activities of the violator, conducting propaganda etc.

(2) Actually, we shall later see that only β need be known to them.

By $\beta < 1$ we mean that the violator prefers breaking to not breaking the treaty, if he is sure that his violation will not be inspected. However, an inspection has a deterrent power, and the violator would change his preference if he is sure that an inspection will follow a violation. This accounts for $0 < \beta$.

Thus, $\gamma = 1-\beta$ measures how important it is to the violator to conduct a secret violation. If γ is near 0 , he is almost indifferent between obeying on the one hand and violating uninspected on the other hand. If, however, it is very important for the violator to conduct a secret test, then γ would be nearly 1 , in which case obeying the treaty would look to him almost as bad as violating it and being inspected.

Needless to say, γ could be greater than or equal to 1, $(\beta < 0)$; i.e., that even a sure inspection will not deter the violator. We do not treat such cases in this paper.

Discussion. A real-life situation is, in general, more complicated. The number of events need not be fixed, nor need they be equally likely suspicious. The number of available inspections need not be fixed. The probability of identifying an inspected violation may vary. There may be different utility gains if, say, an uninspected violation takes place in the first or in the 8th or in the n-th event. The "psychological" aspects may play a decisive role, etc.

Such circumstances can in many cases be included in a more refined model and treated successfully. There are other instances in which a refinement is not so easy to make. For instance, the real-life situation may not be limited to just n events. It may perhaps resemble an infinite sequence of games of our type, with utilities which vary from game to game, and which depend on the outcomes of the previous games. It is not a priori clear that what is a good strategy for each game alone is also a good strategy for the entire situation.

Possibly both players have their own duties, each is allowed to inspect the other side. Such a 2-sided inspectors' game offers more mutual agreements prior to and during the playing of the game, and it may be an essentially different game.

Thus, it should be stressed that applying our model or its refinements to a real-life situation should not be made without a careful consideration.

Measuring the utilities α and β is not an easy undertaking. In many cases it is absurd to assume that the utilities are known to both players. We shall have something to say on this in this connection in Section 8.

It follows from Assumption 5, that the inspector will plan to use all his available inspections(1). He has, therefore, $\binom{n}{r}$ pure strategies, each of which consists of a choice of r out of the n events, to be inspected. His mixed strategy is a probability distribution of these strategies.

It will be convenient to furnish a different description for such mixed strategies, which describes prior to each event $k + 1$, $k = 0, 1, 2, \ldots, n - 1$, the probability of inspecting it, when all the necessary information on the previous inspections is given. Accordingly, we make the following definition:

A history of k events, $1 \leq k \leq n - 1$, denoted generically by $h(k)$, is the set of those events, among the first k events, that were actually inspected. Clearly, $h(k)$ is known to both parties after the k events had already taken place. We define $h(o)$ to be the empty set.

A possible history of k events is any history that could have taken place under the rules of the game. Thus, it is a subset of the first k events, $0 \leq k \leq n$, which contains at most r elements,

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(1) Of course, the moment a violation occurs, the game may be considered over, prior to the next event and more inspections are irrelevant. But there is no loss of generality in assuming that the inspector will always act as if no violation has previously occurred.

and, moreover, if $k = n-t$ and $1 \leq t \leq r-1$, then the history must have contained at least $r-t$ elements.

It is easy to see that any mixed strategy for the inspector can be described as the set $\{q\} = \{q(h(k))\}$, $0 \leq q \leq 1$, where $q(h(k))$ is the probability of inspecting the event $k+1$, if the history $h(k)$ has taken place previously. $q(h(k))$ is defined for all the possible histories of k events and $\{q\}$ is a strategy for the inspector if and only if

(2.2) $\quad q(h(k)) = 0$, whenever $h(k)$ contains r elements,

(2.3) $\quad q(h(k)) = 1$, whenever $k = n-t$, $1 \leq t \leq r$ and $h(k)$ has exactly $r-t$ elements.

If $q(h(k))$ depends only on the number of elements in $h(k)$, for each possible history, then $\{q\}$ is called a strongly behavioral strategy.

A pure strategy for the violator is a set $\{\ell\} = \{\ell(h(k))\}$, where $\ell = 0, 1$ is defined for each possible history and denotes whether $(\ell = 0)$ or not $(\ell = 1)$ the event $k+1$ is to be violated. It must satisfy:

(i) If

(2.4) $$\ell(h_1(k_1)) = 1, \quad h_1(k_1) \subset h_2(k_2), \quad k_1 < k_2,$$

then

(2.5) $$\ell(h_2(k_2)) = 0.$$

Indeed, only one violation can take place Assumption 2.

(ii) If $k = n - t$, $1 \leq t \leq r$, and $h(k)$ has $r-t$ elements then

$$\ell(h(k)) = 0 \quad . \tag{2.6}$$

Indeed, under these conditions the violator knows that the event $k+1$ will be inspected, and therefore, by Assumption 6 he will not choose this event for a violation.

(iii) If $h(k)$ contains r elements, and the k-th event is among them, then Assumption 6 predicts that a violation must occur on one of the events $k + 1$, $k + 2$, ..., n, unless an inspection has occurred previously. In order not to bother ourselves with sets of equivalent strategies (See Assumption 4), we shall henceforth decide that under such circumstances, the $(k + 1)$-th event will be inspected.

Thus, except for strategies which are equivalent to the following ones, the totality of pure strategies which are available to the violator consists of those which satisfy (i), (ii) and (iii) above.

The inspector game $\Gamma(n, r)$ is a 2-person game which satisfies the above assumptions, and its strategies satisfy the above requirements.

We impose the following additional assumptions on its players:

Assumption 7. The Violator cannot announce his choice of pure or mixed strategy.

In fact, he has just signed a treaty which he promised to fulfill.

Assumption 8. The inspector has the option of announcing and committing himself to a specific mixed strategy, prior to playing the game. If he uses his option, the violator hears the announcement and knows that it will be observed.

This is usually the case in real-life situations, for people would not risk their reputation by not fulfilling their commitments in a way which is easy to discover. To make sure that the violator believes him, the inspector may agree that an objective outsider will turn the roulette wheel.

Assumption 9. Each player is rational; i.e., of any two alternatives which give rise to outcomes, a player will choose the one which yields him the higher expected payoff.

3. Optimal Strategies.

We shall now study the consequences if the inspector uses his option of Assumption 8. We shall prove in Section 6, that, to say the least, using this option will not harm his interests essentially.

Using Assumption 9, we shall now describe possible goals of the players, and we shall later show that these goals can be achieved.

Procedure A. For any announced[1] strategy of the inspector, the violator will choose his (pure) strategy which maximizes his expected payoff. If several such strategies are available to the violator, he will adopt a strategy among them which will MAXIMIZE the inspector's expected payoff. Knowing this, the inspector will announce that strategy which will maximize his expected payoff. If the inspector has several choices, he will choose a strategy among them which will minimize[2] the violator's expected payoff.

Remark. It seems strange that we require that the inspector counts on the good will of the violator, who, having achieved his maximum payoff is supposed to also seek the benefit of the inspector. We shall see in Section 4 that only slight changes occur if this is not required.

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(1) From now on, it should be understood that "announced" means "announced and committed to".

(2) One could just as well replace "minimize" by "maximize", or by any other rule which leads to a determined payoff.

Theorem 3.1. Let Γ be an arbitrary 2-person game, where the players, called the violator and the inspector (1) are subject to Assumptions 7 and 8. There always exists a solution to the goals of Procedure A. It yields the players a unique payoff.

Proof. Let $1, 2, \ldots, s$ be the pure strategies of the inspector, then each of his mixed strategies is a probability distribution $\underline{p} = (p_1, p_2, \ldots, p_s)$ which can be regarded as a point in the simplex S :

(3.1) $$p_1 + p_2 + \cdots + p_s = 1 , \quad p_i \geq 0 , \quad i = 1, 2, \ldots, s .$$

In a similar fashion we can describe each strategy for the violator as a point in a simplex T.

For each announced strategy $\underline{p}$ by the inspector, the violator has at least one pure strategy which yields him a maximum expected payoff, since he has a finite number of pure strategies. Therefore, the set of (mixed) strategies for the violator which yields him a maximum expected payoff is a face T^* of T. Obviously, he can choose from these strategies one which yields a maximum expected payoff to the inspector, since the payoff to the inspector varies linearly with the

(1) These players do not necessarily assume the roles imposed by Assumptions 1-6.

strategy used by the violator. Moreover, he can even choose a <u>pure</u> strategy which yields him and the violator these payoffs. We shall assume that he only chooses a pure strategy, and this will entail no loss of generality.

Thus, for each announced strategy $\underline{p}$, Procedure A dictates unique payoffs $f_I(\underline{p})$ and $f_V(\underline{p})$ to the inspector and to the violator, respectively.

Let

$$v^* = \overline{\lim}\ f_I(\underline{p}) \ , \quad \underline{p} \in S . \tag{3.2}$$

There exists an infinite sequence of strategies $\underline{p}^{(1)}, \underline{p}^{(2)}, \ldots,$ converging to a point $\underline{p}^{(o)} \notin S$, such that

$$\lim_{n \to \infty} f_I(\underline{p}^{(n)}) = v^* . \tag{3.3}$$

Moreover, since the violator has only a finite number of pure strategies, we can assume that he uses the same pure strategy ξ in answer to each of the strategies $\underline{p}^{(1)}, \underline{p}^{(2)}, \ldots$. Strategy ξ need not be the one chosen as an answer to $\underline{p}^{(o)}$, but at any rate,

$$f_V(\underline{p}^{(o)}) = \lim_{n \to \infty} f_V(\underline{p}^{(n)}) \tag{3.4}$$

$$f_I(\underline{p}^{(o)}) \geq \lim_{n \to \infty} f_I(\underline{p}^{(n)}) = v^* . \tag{3.5}$$

Indeed, if (3.4) were not true, then the left hand side would be greater than the right hand side, since by employing ξ as an answer to $\underline{p}^{(o)}$, the violator can guarantee himself the right hand side. Let ξ^* be the strategy employed by the violator as an answer to $\underline{p}^{(o)}$, then ξ^* employed against $\underline{p}^{(n)}$ would yield him a payoff which is greater than $f_V(\underline{p}^{(n)})$, for a sufficiently large n. This contradicts the definition of $f_V(\underline{p})$. Relation (3.5) is certainly true, since, as we have just shown, ξ as an answer to $\underline{p}^{(o)}$ yields the violator and the inspector the expected payoffs $f_V(\underline{p}^{(o)})$ and v^*, respectively, and the violator is assumed to be seeking the welfare of the inspector in addition.

There is a finite number of points $\underline{p}$ in S for which $f_I(\underline{p}) > v^*$. If no such point exists, then $v = \text{Max}_{\underline{p} \in S} f_I(\underline{p}) \geq v^*$ exists, and is achieved at $\underline{p}^{(o)}$. If this number is not zero, then v exists and is achieved either at $\underline{p}^{(o)}$ or somewhere else.

Let K be the set of points $\underline{p}$ such that $f_I(\underline{p}) = v$, $\underline{p} \in S$. We shall show that K is compact. If $\hat{\underline{p}}^{(1)}, \hat{\underline{p}}^{(2)}, \ldots$ is an infinite sequence of points in K, converging to a point $\hat{\underline{p}}^{(o)}$, we have to show that $\hat{\underline{p}}^{(o)} \in K$. Again, we may assume that the violator answers each $\hat{\underline{p}}^{(i)}$, $i = 1, 2, \ldots$, by the same pure strategy η. The same reasoning which lead to (3.4) and to (3.5) yields

(3.6) $$f_V(\hat{\underline{p}}^{(o)}) = \lim_{n\to\infty} f_V(\hat{\underline{p}}^{(n)}) \quad ,$$

(3.7) $$f_I(\hat{\underline{p}}^{(o)}) \geq \lim_{n\to\infty} f_I(\hat{\underline{p}}^{(n)}) = v \quad .$$

By definition, $f_I(\hat{\underline{p}}^{(o)}) \leq v$; hence $f_I(\hat{\underline{p}}^{(o)}) = v$, or $\hat{\underline{p}}^{(o)} \in K$.

By the compactness of K, it follows readily that there exists a point in K, for which the violator gets the lowest expected payoff. This completes the proof.

Remark. From now on we shall restrict the violator to pure strategies, since he cannot gain in his goals by using mixed strategies.

A pair of strategies which adheres to Procedure A will be called a pair of optimal strategies. A strategy for the inspector will be called optimal, if there exists a strategy for the violator such that the pair of strategies is optimal.

We return to the game $\Gamma(n, r)$. Let $\{q_{n,r}\} \equiv \{q_{n,r}(h(k))\}$ and $\{\ell_{n,r}\} \equiv \{\ell_{n,r}(h(k))\}$ be a fixed pair of optimal strategies for this game, as described in Section 2. We shall be interested in $q^o_{n,r} \equiv q_{n,r}(h(o))$ and $\ell^o_{n,r} \equiv \ell_{n,r}(h(o))$; i.e., in the probabilities of inspecting and violating the first event, respectively, as determined by these strategies[1].

[1] $\ell^o_{n,r}$ can be either 0 or 1 .

After the "roulette wheel" has been spun, and all the decisions concerning the first event have been made, there can be one and only one of four consequences, as shown in Table 1.

	$\ell^o_{n,r} = 1$	$\ell^o_{n,r} = 0$
The roulette wheel calls for inspection	A violation occurred and it was inspected	The parties face a game $\Gamma(n-1, r-1)$
The roulette wheel calls for no inspection	A violation occurred and it was not inspected	The parties face a game $\Gamma(n-1, r)$

Table 1 .

If $\ell^o_{n,r} = 1$, the game may be considered over after the first event, since the payoff is independent of the actions taken in further moves. Consequently, although, in general, the inspector does not know whether $\ell^o_{n,r} = 0$ or 1 , it would not harm him to assume that $\ell^o_{n,r} = 0$.

Theorem 3.2. Let $q^o_{n^*, r^*}$ and $\ell^o_{n^*, r^*}$ be the specifications for the first move, generated by fixed pairs of optimal strategies $\{q_{n^*, r^*}\}$ $\{\ell_{n^*, r^*}\}$, for the games $\Gamma(n^*, r^*)$, $0 \leq r^* \leq n^* \leq n$. Then, the following is an optimal strategy for $\Gamma(n, r)$: the inspector inspects the first event with a probability $q^o_{n,r}$. If he has inspected the first event, he will inspect the second event with a probability $q^{o}_{n-1, r-1}$, if not - he will inspect the second event with a probability $q^{o}_{n-1, r}$; and so on. At each stage he determines the number n^{*} of events still left and the number r^{*} of inspections still available, and inspects the coming event with the probability $q_{n^{*}, r^{*}}$. The violator chooses the first event for violation if $\ell^{o}_{n, r} = 1$. If not, he violates the second event either if an inspection occurred in the first event and $\ell^{o}_{n-1, r-1} = 1$ or if an inspection did not occur in the first event and $\ell^{o}_{n-1, r} = 1$ and so on. At each stage, the violator chooses to violate the coming event if no violation occurred previously, and if $\ell^{o}_{n^{*}, r^{*}} = 1$, where n^{*} and r^{*} are, the number of events and available inspections still left, respectively.

Proof. Clearly, $q^{o}_{n^{*}, o} = 0$, $q^{o}_{n^{*}, n^{*}} = 1$, $\ell^{o}_{n^{*}, o} = 1$, $\ell^{o}_{n^{*}, n^{*}} = 0$, $n^{*} = 1, 2, \ldots, n$, therefore this strategy satisfies all the requirements of Section 2. The theorem is true for $n=1$, $r=0$ or 1, since for each case there is only one pair of strategies and it is optimal. Let us assume that the theorem is true for all the games $\Gamma(n^{*}, r^{*})$, where $0 \leq r^{*} \leq n^{*} < n$. Clearly, the theorem is true for $\Gamma(n, n)$, so we have to check only the games $\Gamma(n, r)$ for $0 \leq r < n$. If $\ell^{o}_{n, r} = 1$, then the game is over after the first event and the pair of strategies described in the theorem is optimal because its relevant part coincides with the optimal strategies $\{q_{n, r}\}$ and $\{\ell_{n, r}\}$.

We now assume that $\ell^{o}_{n,r} = 0$, in which case, the players will face either $\Gamma(n-1, r-1)$ or $\Gamma(n-1, r)$ in the second event, with probabilities $q^{o}_{n,r}$ and $1-q^{o}_{n,r}$, respectively.

The pair $\{q_{n,r}\}$ and $\{\ell_{n,r}\}$ is optimal for $\Gamma(n,r)$. It yields the expected payoffs $v_{n,r}$ and $w_{n,r}$ to the inspector and to the violator, respectively. We shall show that the strategy described in the theorem yields the same expected payoff. The strategies $\{q_{n,r}\}$ $\{\ell_{n,r}\}$ specify, in particular, what actions should be taken from the second event on, in each of the two possibilities. Let us, therefore, denote by $\{q^{*}_{n-1,r-1}\}$ and $\{\ell^{*}_{n-1,r-1}\}$ the generated strategies when the players face $\Gamma(n-1, r-1)$ and by $\{q^{*}_{n-1,r}\}$ and $\{\ell^{*}_{n-1,r}\}$ the generated strategies when the players face[(1)] $\Gamma(n-1, r)$. Let $v^{*}_{n-1,r-1}$ and $w^{*}_{n-1,r-1}$ be the expected payoffs to the inspector and to the violator, respectively, if the strategies $\{q^{*}_{n-1,r-1}\}$ and $\{\ell^{*}_{n-1,r-1}\}$ are used in a game $\Gamma(n-1, r-1)$. Similarly, let $v^{*}_{n-1,r}$ and $w^{*}_{n-1,r}$ be the expected payoffs if the game is $\Gamma(n-1, r)$. Clearly,

(3.8) $$v_{n,r} = q^{o}_{n,r}\, v^{*}_{n-1,r-1} + (1-q^{o}_{n,r})\, v^{*}_{n-1,r} ,$$

(3.9) $$w_{n,r} = q^{o}_{n,r}\, w^{*}_{n-1,r-1} + (1-q^{o}_{n,r})\, w^{*}_{n-1,r} .$$

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(1) If $q^{o}_{n,r} = 0$ or 1, $\{q_{n,r}\}$ and $\{\ell_{n,r}\}$ need not specify anything for the game which occurs with a zero probability; however, in this case we are free to choose any strategy for such a game. We shall therefore define
$\{q^{*}_{n-1,r-1}\} = \{q_{n-1,r-1}\}$ if $q^{o}_{n,r} = 0$ and $\{q^{*}_{n-1,r}\} = \{q_{n-1,r}\}$ if $q^{o}_{n,r} = 1$.

Tracing Procedure A , we shall show that the starred strategies are optimal in their corresponding games. Indeed, if the inspector could get more in either $\Gamma(n-1, r-1)$ or in $\Gamma(n-1, r)$, by employing a different strategy, the corresponding game would have occurred with a non-zero probability (since otherwise, by the preceding footnote, the starred strategy is optimal). By (3.8), the inspector could have done better than an expected payoff $v_{n,r}$, in the original game, contrary to the fact that $\{q_{n,r}\}$ is an optimal strategy. A similar contradiction results from (3.9), if a strategy different from $\{\ell^*_{n-1,r-1}\}$ or $\{\ell^*_{n-1,r}\}$ could be used to benefit the violator in the corresponding game. Finally, by (3.8) or (3.9), a contradiction would result if, either the violator can replace a starred strategy in the corresponding game by a strategy which gives him the same expected payoff but increases the expected payoff to the inspector, or if the inspector can replace a starred strategy in the corresponding game by a strategy which gives him the same expected payoff but decreases the expected payoff to the violator.

By the induction hypothesis, the strategy pair described by the theorem generates optimal strategy pairs for the games $\Gamma(n-1, r-1)$ and $\Gamma(n-1, r)$ in case either occurs on the second move with a non-zero probability; therefore, these generated strategies yield the expected payoffs $v^*_{n-1,r-1}$ $w^*_{n-1,r-1}$ and $v^*_{n-1,r}$ $w^*_{n-1,r}$ to whichever game arises.

Therefore, by (3.8) and (3.9) the strategy pair described by the theorem yields the expected payoffs $v_{n,r}$, $w_{n,r}$ in the original game. Consequently, this is a pair of optimal strategies in $\Gamma(n,r)$.

Remark. We have proved, moreover, that as long as $q^{o}_{n*,r*} \neq 0,1$ and as long as no violation occurs, the optimal strategies must proceed according to the description in the theorem; i.e., at each stage which occurs with a non-zero probability, the "rest" of the strategy pair must be optimal for the coming events regarded as a new game, provided that no violation has occurred previosuly.

Corrolary 3.1. The game $\Gamma(n,r)$ can be solved, according to the goals of Procedure A, by strongly behavioral strategies [1].

Proof. The strategies described in Theorem 3.2 are strongly behavioral.

Theorem 3.2 will be used for constructing optimal strategies and for arriving at recursion formulas concerning the expected payoffs.

Clearly, if $r=n$, there is only one strategy pair, namely, to inspect each event and never to violate the treaty. Also, if $r=0$,

(1) I.e., by strategies which, at each stage depend only on the number of the events and the available inspections still left.

the inspector cannot inspect, and the violator will violate the first event. Both strategy pairs are optimal, and

(3.10) $\quad v_{n,n} = \alpha \quad w_{n,n} = 1-\gamma, \quad n = 1, 2, \ldots, \quad \gamma = 1-\beta,$

(3.11) $\quad v_{n,o} = 0 \quad w_{n,o} = 1, \quad n = 1, 2, \ldots \quad .$

Henceforth, we shall assume that $0 < r < n, \quad n = 2, 3, \ldots$.

Our procedure will be similar to the method employed by M. Dresher [1]. The same procedure has been used (1) by H. W. Kuhn [3].

We shall derive the recursion formulas by analyzing Table 2, which is justified by Theorem 3.2.

		Violator: Violate first event	Violator: Do not violate first event
$q^o_{n,r}$	Inspector: inspect first event	1, 0	$v_{n-1,r-1}$, $w_{n-1,r-1}$
$1-q^o_{n,r}$	Inspector: Do not inspect first event	0, 1	$v_{n-1,r}$, $w_{n-1,r}$

Payoff Matrix.

Table 2 .

(1) Both papers treat constant-sum games.

I. If the announced inspector's strategy is optimal from the second event on, and if

$$1-q^o_{n,r} > q^o_{n,r} \; w_{n-1,r-1} + (1-q^o_{n,r}) \; w_{n-1,r} \; , \tag{3.12}$$

then the violator should choose to inspect the first event, since his expected payoff would be maximal with this choice. The expected payoffs will be $f_V(q^o_{n,r}) = 1-q^o_{n,r}$ and $f_I(q^o_{n,r}) = q^o_{n,r}$, to the violator and to the inspector, respectively.

II. If, instead of (3.12),

$$1-q^o_{n,r} < q^o_{n,r} \; w_{n-1,r-1} + (1-q^o_{n,r}) \; w_{n-1,r} \; , \tag{3.13}$$

then the violator should not inspect the first event. The expected payoffs will then be $f_V(q^o_{n,r}) = q^o_{n,r} \; w_{n-1,r-1} + (1-q^o_{n,r}) \; w_{n-1,r}$

and $\qquad f_I(q^o_{n,r}) = q^o_{n,r} \; v_{n-1,r-1} + (1-q^o_{n,r}) \; v_{n-1,r}$.

III. If, instead of (3.12),

$$1-q^o_{n,r} = q^o_{n,r} \; w_{n-1,r-1} + (1-q^o_{n,r}) \; w_{n-1,r} \; , \tag{3.14}$$

the violator will have the expected payoff $f_V(q^o_{n,r}) = 1-q^o_{n,r}$, regardless of his choice. However, by Procedure A, he will choose that strategy which maximizes the inspector's payoff; hence

$$f_I(q^o_{n,r}) = \mathrm{Max}(q^o_{n,r}, \; q^o_{n,r} v_{n-1,r-1} + (1-q^o_{n,r}) \; v_{n-1,r}) \; .$$

Denote

$$(3.15)\qquad t_{n,r} = \frac{1-w_{n-1,r}}{1-w_{n-1,r}+w_{n-1,r-1}}\ , \quad 0 < r < n\ ,$$

$$t_{n,o} \equiv 0\ , \quad t_{n,n} \equiv \gamma\ , \quad n = 1, 2, \ldots\ .$$

We shall later prove that the denominator cannot vanish, since $w_{n-1,r} < w_{n-1,r-1}$. It will then follow that $0 < t_{n,r} < 1$, $0 < r < n$.

Remembering that the inspector is interested in obtaining the highest expected payoff to himself, we compute and compare his highest expected payoffs in each of the above cases:

$$(3.16)\qquad \sup_{q^o_{n,r} < t_{n,r}} f_I(q^o_{n,r}) = \sup_{q^o_{n,r} < t_{n,r}} q_{n,r} = t_{n,r}\ .$$

$$(3.17)\qquad \sup_{q^o_{n,r} > t_{n,r}} f_I(q^o_{n,r}) = \sup_{q^o_{n,r} > t_{n,r}} [q^o_{n,r}\, v_{n-1,r-1} + (1-q^o_{n,r})v_{n-1,r}] =$$

$$= t_{n,r} v_{n-1,r-1} + (1-t_{n,r})\, v_{n-1,r}\ ,$$

since, obviously[1], $v_{n-1,r} \geq v_{n-1,r-1}$.

---

(1) For the same number of events, the inspector has more available inspections in $\Gamma(n-1, r)$.

Finally,

$$f_I(q^o_{n,r}) = \text{Max}[t_{n,r},\ t_{n,r} v_{n-1,r-1} + (1-t_{n,r}) v_{n-1,r}] , \tag{3.18}$$

if $q^o_{n,r} = t_{n,r}$.

Thus, the highest expected payoff $v_{n,r}$ is obtained if $q^o_{n,r} = t_{n,r}$, and it is equal to the right hand side of (3.18).

Clearly, the supremum in (3.16) is not achieved. Had we known that $v_{n-1,r} > v_{n-1,r-1}$, we would have deduced that the supremum in (3.17) is not achieved. This would prove that only case III must be considered for Procedure A, that

$$q^o_{n,r} = t_{n,r} ,\quad 0 < r < n ,\quad q^o_{n,o} = 0 ,\ q^o_{n,n} = 1 ,\quad n = 1, 2, \ldots , \tag{3.19}$$

and that

$$w_{n,r} = 1-t_{n,r} ,\quad 0 \leq r \leq n ,\quad n = 1, 2, \ldots . \tag{3.20}$$

(Compare (3.10) and (3.11) and the section preceding these formulas for the cases $r = 0$, $r = n$) . The following lemma will show that this is, indeed, the case.

Lemma 3.1 . If $0 < r < n$, then $w_{n-1,r} < w_{n-1,r-1}$ and $v_{n-1,r} > v_{n-1,r-1}$.

Proof. By (2.1), (3.10) and (3.11), the lemma holds for $n = 2$. Suppose that the lemma is true for $n = \nu$, $\nu \geq 2$, we shall prove it for $n = \nu+1$.

a. By (2.1) (3.10) and the induction hypothesis, $0 < 1-\gamma = \beta =$ $= w_{\nu-1,\nu-1} < w_{\nu-1,\nu-2}$; hence, by (3.15) and (2.1), $0 < t_{\nu,\nu-1} = \gamma/(\gamma + w_{\nu-1,\nu-2}) < \gamma < 1 < \alpha$. Consequently, by (3.18), and the lines which follow this formula,

$$0 < v_{\nu,\nu-1} = \mathrm{Max}[\, t_{\nu,\nu-1},\ t_{\nu,\nu-1} v_{\nu-1,\nu-2} + (1-t_{\nu,\nu-1}) v_{\nu-1,\nu-1}\,] < \alpha ,$$

because, by the induction hypothesis and (3.10), $v_{\nu-1,\nu-2} < v_{\nu-1,\nu-1} = \alpha$. Thus, by (3.10), $0 < v_{\nu,\nu-1} < v_{\nu,\nu} = \alpha$. Also, since $v_{\nu-1,\nu-2} < v_{\nu-1,\nu-1}$, we can apply (3.20) and deduce that $w_{\nu,\nu-1} = 1-t_{\nu,\nu-1}$; hence, by (3.10), $\beta = w_{\nu,\nu} < w_{\nu,\nu-1}$, because $1-t_{\nu,\nu-1} > 1 - \gamma = \beta$. This proves the lemma for $n = \nu + 1$, $r = \nu$.

b. By the induction hypothesis and (3.11), $w_{\nu-1,1} < w_{\nu-1,o} = 1$; hence, $0 < t_{\nu,1} = (1 - w_{\nu-1,1})/(2 - w_{\nu-1,1}) < 1$. Consequently, by (3.18), $0 < v_{\nu,1} = \mathrm{Max}[\, t_{\nu,1}\ t_{\nu,1}\ v_{\nu-1,o} + (1 - t_{\nu,1})\ v_{\nu-1,1}]$; hence, $0 = v_{\nu,o} < v_{\nu,1}$. Moreover, since, by the induction hypothesis, $v_{\nu-1,1} > v_{\nu-1,o} = 0$, we can apply (3.20) and deduce that $w_{\nu,1} = 1 - t_{\nu,1}$. Thus, $1 = w_{\nu,o} < w_{\nu,1}$. This proves the lemma for $n = \nu + 1$, $r = 1$.

c. There remains the case $n = \nu+1$, $1 < r < \nu$. In this case by (3.18),

$$v_{\nu,r} = \mathrm{Max}[t_{\nu,r},\ t_{\nu,r}\ v_{\nu-1,r-1} + (1 - t_{\nu,r})\ v_{\nu-1,r}], \tag{3.21}$$

$$(3.22) \qquad v_{\nu, r-1} = \text{Max}[t_{\nu, r-1},\ t_{\nu, r-1} v_{\nu-1, r-2} + (1-t_{\nu, r-1}) v_{\nu-1, r-1}] .$$

It is easy to verify from (3.15) and from the induction hypothesis that

$$(3.23) \qquad 0 < \beta \leq w_{\nu-1, r} < w_{\nu-1, r-1} < w_{\nu-1, r-2} \leq 1 , \quad 1 < r < \nu ,$$

$$(3.24) \qquad 0 < t_{\nu, r-1} < t_{\nu, r} < 1 , \qquad 1 < r < \nu ,$$

$$(3.25) \qquad v_{\nu-1, r-2} < v_{\nu-1, r-1} < v_{\nu-1, r} , \qquad 1 < r < \nu \ ;$$

therefore, $v_{\nu, r} > v_{\nu, r-1}$ and[1] by (3.20), $w_{\nu, r} < w_{\nu, r-1}$. This completes the proof of the lemma.

Incidentally, it follows from $\beta = w_{n, n} < w_{n, r} < w_{n, o} = 1$, $0 < r < n$, that

$$(3.26) \qquad 0 < t_{n, r} < \gamma = 1 - \beta < 1 , \quad 0 < r < n .$$

It remains to find out which expression yields the maximum in (3.18). By (3.20) and (3.15), it follows that $t_{n, r}$ satisfies the recursion formula:

$$(3.27) \qquad t_{n, r} = \frac{t_{n-1, r}}{1 + t_{n-1, r} - t_{n-1, r-1}} , \quad 0 < r < n, \quad n = 2, 3, \ldots .$$

(1) (3.20) is used since it is substantiated by the induction hypothesis.

Lemma 3.2. If $0 \leq r \leq n$, then $v_{n,r} \geq t_{n,r}$, and strict inequality holds for $0 < r \leq n$, $n = 1, 2, \ldots$.

Proof. The lemma certainly holds if $r = 0$ and if $r = n$. In particular, therefore, it holds for $n = 1$. Suppose it holds for $n = \nu-1, \nu = 2, 3, \ldots$; we shall show that it holds for $n = \nu$ $0 < r < \nu$. Using (3.18), (3.26), (3.27) and the induction hypothesis, we find that

$$(3.28) \qquad v_{\nu,r} \geq t_{\nu,r} v_{\nu-1,r-1} + (1-t_{\nu,r})v_{\nu-1,r} >$$

$$t_{\nu,r} t_{\nu-1,r-1} + t_{\nu-1,r} - t_{\nu,r} t_{\nu-1,r} =$$

$$= \frac{t_{\nu-1,r}(t_{\nu-1,r-1} - t_{\nu-1,r})}{1 + t_{\nu-1,r} - t_{\nu-1,r-1}} + t_{\nu-1,r} =$$

$$= \frac{t_{\nu-1,r}}{1 + t_{\nu-1,r} - t_{\nu-1,r-1}} = t_{\nu,r} \; .$$

This completes the proof.

It follows from Lemma 3.2, that for $0 < r < n$, the maximum in (3.18) is reached by the second expression on the right hand side.

Combining the above results, we can now state

Theorem 3.3. The expected payoffs $v_{n,r}$ and $w_{n,r}$ for the inspector and the violator, respectively, obtained by using optimal strategies in the game $\Gamma(n, r)$ satisfy the recursion formulas

$$(3.29) \qquad v_{n,r} = w_{n,r}\, v_{n-1,r} + (1-w_{n,r})\, v_{n-1,r-1} , \qquad 0 < r < n ,$$

$$(3.30) \qquad w_{n,r} = \frac{w_{n-1,r-1}}{1 - w_{n-1,r} + w_{n-1,r-1}} , \qquad 0 < r < n ,$$

$$(3.31) \qquad v_{n,o} = 0 , \quad v_{n,n} = \alpha , \quad w_{n,o} = 1 , \quad w_{n,n} = 1-\gamma , \quad \underline{\text{for}} \quad n = 1, 2, \ldots .$$

<u>Proof.</u> Formula (3.31) is a repetition of (3.10) and (3.11). The other formulas follow from (3.20), (3.21), (3.27) and Lemma 3.2.

<u>Theorem 3.4.</u> <u>There is a unique pair of optimal strategies</u> $\{q(h(k))\}$ <u>and</u> $\{\ell(h(k))\}$ <u>to the game</u> $\Gamma(n, r)$, <u>of the type described by Theorem</u> 3.2; <u>namely</u> $q(h(k)) = 1 - w_{n^*, r^*}$ if $0 \leq r^* < n^*$, $q(h(k)) = 1$ if $r^* = n^*$, <u>$\ell(h(k)) = 1$ the first time r^* becomes 0, (if this happens), and</u> $\ell(h(k)) = 0$ <u>otherwise. Here, n^* and r^* denote, respectively, the number of events and available inspections left after the history $h(k)$ takes place</u>.

<u>Proof.</u> By Theorem 3.2, it is sufficient to verify the theorem for the first move. It certainly holds if $r = 0$ or if $r = n$. For $0 < r < n$, $q^o_{n,r}$ must satisfy (3.19); i.e., by (3.20), $q^o_{n,r} = 1-w_{n,r}$. Thus, the first move for the inspector is determined. Moreover, we know that only case III can yield the inspector the highest payoff,

(see the discussion preceding Lemma 3.1). In this case, the violator obtains the same expected payoff, regardless of his strategy choice; but, by Lemma 3.2, and (3.18), only if he chooses not to violate the first event, is the expected payoff to the inspector maximized, as required by Procedure A; hence $\ell^{o}_{n,r} = 0$ for $0 < r < n$. This completes the proof.

<u>Discussion.</u> The optimal strategies can be given the following verbal description:

As long as $0 < r^{*} < n^{*}$, the inspector announces that probability of inspecting the next event, that will render the violator the same expected payoff, regardless of his strategy choice. If r^{*} becomes equal to n^{*}, the inspector inspects each of the subsequent events. If r^{*} becomes 0 and $n^{*} > 0$, the inspector, of course, cannot inspect any additional event.

The violator's strategy is simply to wait until all the inspections are used up, and then inspect if there are events left.

So far we have limited ourselves to optimal strategies of the type described by Theorem 3.2. We are now in a position to show that there are no other pairs of optimal strategies. Indeed, as long as $0 < r^{*} < n^{*}$, and as long as no violation has previously occurred, $q(h(k)) = t_{n^*, r^*}$ lies in the open interval $(0,1)$, by (3.26); hence, by the remark which followed the proof of Theorem 3.2, the next event must be inspected with the probability t_{n^*, r^*} and a violation must not occur for this event. This process is unique, and it will continue until either $r^* = 0$ or $r^* = n^*$, in which case the process will still continue in a unique way. Summarizing this result we state

Corollary 3.2. There is a unique pair of optimal strategies[1].

Theorem 3.5. The optimal strategy used by the inspector renders the violator the same expected payoff, regardless of his strategy choice[2]. It is a minimax strategy for the inspector, for a constant-sum game defined by the violator's payoff matrix alone.

Proof. By induction the theorem is true if $r = 0$ or $r = n$, since in these cases, only one strategy is available to the inspector. In particular, it is true if $n = 1$. Suppose that the first part of the theorem is true for $n-1$ events, $0 \leq r \leq n-1$. Consider the game $\Gamma(n, r)$, $1 \leq r \leq n-1$. In this case, we know that $q^o_{n,r} = t_{n,r}$ (see (3.19)). Let ξ be a pure strategy for the violator, which calls for a violation of

(1) Within the strategy space defined by the requirements in Section 2.

(2) Provided that the violator chooses a strategy which satisfies the requirement.

the first event. Then, if the violator uses this strategy as an answer to the inspector's optimal strategy $\{q_{n,r}\}$, his expected payoff is equal to $1-t_{n,r} = w_{n,r}$ (see (3.20)). Let η be a pure strategy for the violator which does not call for a violation of the first event; then if he uses η as an answer to $\{q_{n,r}\}$, the players will face in the second event the game $\Gamma(n-1, r-1)$ with the probability $t_{n,r}$ and the game $\Gamma(n-1, r)$ with the probability $1-t_{n,r}$. Strategy η generates strategies η_1 and η_2 for the games $\Gamma(n-1, r-1)$ and $\Gamma(n-1, r)$, respectively, which, by the induction hypothesis, yield the violator an expected payoff $w_{n-1, r-1}$ and $w_{n-1, r}$, respectively, because the strategies generated by $\{q_{n,r}\}$ for the corresponding games are optimal. The violator's expected payoff is therefore equal to $t_{n,r} w_{n-1,r-1} + (1-t_{n,r}) w_{n-1,r}$, which, by (3.14), (3.19) and (3.20) is equal to $1-t_{n,r} = w_{n,r}$. Thus, whatever strategy is used by the violator, he obtains the same expected payoff $w_{n,r}$.

In order to show that $\{q_{n,r}\}$ is a minimax strategy, we have to show that the violator can guarantee himself at least $w_{n,r}$, when a constant-sum game $\Delta(n, r)$ based on his payoff matrix is played. By induction, we assume that this is true for the games $\Delta(n-1, r)$, $0 \leq r \leq n-1$. The statement is certainly true for $\Delta(n, o)$ and $\Delta(n, n)$. Consider the game $\Delta(n, r)$, $0 < r < n$. Since $\Delta(n, r)$ is a constant-sum game, we can assume that the inspector announces his minimax strategy $\{\hat{q}_{n,r}\}$ prior to playing the game. If $\hat{q}^{o}_{n,r} \leq t_{n,r}$, the violator may choose to violate the first event, and his expected payoff, $1-\hat{q}^{o}_{n,r}$ is greater than or equal to $1-t_{n,r} = w_{n,r}$. (It is strictly greater than $w_{n,r}$ if $\hat{q}^{o}_{n,r} < t_{n,r}$). If $\hat{q}^{o}_{n,r} > t_{n,r}$, the violator may choose not to violate the first event, and from the second event on to act in such a way as to guarantee to himself at least $w_{n-1,r-1}$ if the first event has been inspected, and $w_{n-1,r}$ if not. This is possible by the induction hypothesis. Thus, the violator can guarantee to himself an expected payoff not smaller than $\hat{q}^{o}_{n,r} w_{n-1,r-1} + (1-q^{o}_{n,r})w_{n-1,r}$, which in view of Lemma 3.1 is greater than $t_{n,r} w_{n-1,r-1} + (1-t_{n,r})w_{n-1,r} = 1-t_{n,r} = w_{n,r}$, by (3.14), (3.19) and (3.20). This completes the proof.

4. <u>A non-generous opponent.</u>

It has been required in Procedure A that, after achieving his maximum payoff, the violator's secondary goal is to attempt to maximize the inspector's expected payoff.

Obviously, if this requirement is not made, the inspector will, in general, receive less than $v_{n,r}$ if he announces his optimal strategy (see Lemma 3.2 and (3.18)). If, for instance, the violator's secondary goal is to minimize the inspector's expected payoff, it is easy to see that the extremum problem for the inspector has no solution; (i.e., the supremum of $f_I(\underline{p})$ is not achieved) for $0 < r < n$.

However, we shall show that no matter what the violator's secondary goal is, the inspector can force an outcome arbitrarily close to $(v_{n,r}, w_{n,r})$ by acting properly.

Procedure B. For any announced strategy of the inspector in the game $\Gamma(n, r)$, the violator chooses a strategy which maximizes his expected payoff. If he has several alternatives, he will choose any one of them, either arbitrarily or not. The inspector's goal is to force an outcome in the range[1] $([v_{n,r} - \varepsilon, v_{n,r}], [w_{n,r}, w_{n,r} + \varepsilon])$. Here ε is an arbitrarily small positive number.

Theorem 4.1. The inspector's goal in Procedure B can be achieved if he modifies his optimal strategy[2] $\{q_{n,r}\}$ by slightly increasing $q_{n,r}(h(k))$ whenever $0 < r^* < n^*$. Here, n^* and r^* are, respectively

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(1) Obviously, he cannot expect to obtain more than $v_{n,r}$, nor can he expect that the violator will receive less than $w_{n,r}$, since his strategy is assumed to be announced.

(2) Optimal in the original sense, based on Procedure A.

<u>the number of the events and the number of the available inspections left after history h(k) has taken place.</u>

<u>Proof.</u> The theorem is true for $r=0$ or $r=n$. Suppose that $0 < r < n$. It follows readily by induction and by the analysis in the proof of Theorem 3.2 that when such a modified strategy is announced, then

(i) The violator will choose not to violate until all the available inspections were used up, and then inspect, if there are events left.

(ii) The violator's expected payoff $\hat{w}_{n,r}$ will be greater than $w_{n,r}$.

(iii) The inspector's expected payoff will be less than $v_{n,r}$.

By (i) and Theorem 3.4, we see that the violator's strategy is identical with $\{\ell_{n,r}\}$; therefore, by modifying $\{q_{n,r}\}$ slightly enough, the expected payoffs will lie in the stated intervals.

Thus, from a practical point of view, we are justified in considering Procedure A; and regarding $v_{n,r}$ and $w_{n,r}$ as the final expected payoffs. One need only keep in mind that the optimal strategies should eventually be slightly modified and that the resulting payoffs will only approximate $v_{n,r}$ and $w_{n,r}$.

In Section 9 we shall discuss some other relevant practical aspects.

5. Formulas and Properties of the Expected Payoffs.

The recursion formulas (3.29), (3.30) and (3.31) enable us to compute $w_{n,r}$ and $v_{n,r}$ to any desired pair (n, r). They are tabulated for $1 \leq n \leq 6$ in Tables 3 and 4.

n \ r	0	1	2	3	4	5	6
1	1	$1-\gamma$					
2	1	$\frac{1}{1+\gamma}$	$1-\gamma$				
3	1	$\frac{1+\gamma}{1+2\gamma}$	$\frac{1}{1+\gamma+\gamma^2}$	$1-\gamma$			
4	1	$\frac{1+2\gamma}{1+3\gamma}$	$\frac{1+\gamma+\gamma^2}{1+2\gamma+3\gamma^2}$	$\frac{1}{1+\gamma+\gamma^2+\gamma^3}$	$1-\gamma$		
5	1	$\frac{1+3\gamma}{1+4\gamma}$	$\frac{1+2\gamma+3\gamma^2}{1+3\gamma+6\gamma^2}$	$\frac{1+\gamma+\gamma^2+\gamma^3}{1+2\gamma+3\gamma^2+4\gamma^3}$	$\frac{1}{1+\gamma+\gamma^2+\gamma^3+\gamma^4}$	$1-\gamma$	
6	1	$\frac{1+4\gamma}{1+5\gamma}$	$\frac{1+3\gamma+6\gamma^2}{1+4\gamma+10\gamma^2}$	$\frac{1+2\gamma+3\gamma^2+4\gamma^3}{1+3\gamma+6\gamma^2+10\gamma^3}$	$\frac{1+\gamma+\gamma^2+\gamma^3+\gamma^4}{1+2\gamma+3\gamma^2+4\gamma^3+5\gamma^4}$	$\frac{1}{1+\gamma+\gamma^2+\gamma^3+\gamma^4+\gamma^5}$	$1-\gamma$

The expected payoffs $w_{n,r}$ to the violator

Table 3.

n \ r	0	1	2	3	4	5	6
1	0	α					
2	0	$\frac{\alpha}{1+\gamma}$	α				
3	0	$\frac{\alpha}{1+2\gamma}$	$\frac{\alpha(1+\gamma)}{1+\gamma+\gamma^2}$	α			
4	0	$\frac{\alpha}{1+3\gamma}$	$\frac{\alpha(1+2\gamma)}{1+2\gamma+3\gamma^2}$	$\frac{\alpha(1+\gamma+\gamma^2)}{1+\gamma+\gamma^2+\gamma^3}$	α		
5	0	$\frac{\alpha}{1+4\gamma}$	$\frac{\alpha(1+3\gamma)}{1+3\gamma+6\gamma^2}$	$\frac{\alpha(1+2\gamma+3\gamma^2)}{1+2\gamma+3\gamma^2+4\gamma^3}$	$\frac{\alpha(1+\gamma+\gamma^2+\gamma^3)}{1+\gamma+\gamma^2+\gamma^3+\gamma^4}$	α	
6	0	$\frac{\alpha}{1+5\gamma}$	$\frac{\alpha(1+4\gamma)}{1+4\gamma+10\gamma^2}$	$\frac{\alpha(1+3\gamma+6\gamma^2)}{1+3\gamma+6\gamma^2+10\gamma^3}$	$\frac{\alpha(1+2\gamma+3\gamma^2+4\gamma^3)}{1+2\gamma+3\gamma^2+4\gamma^3+5\gamma^4}$	$\frac{\alpha(1+\gamma+\gamma^2+\gamma^3+\gamma^4)}{1+\gamma+\gamma^2+\gamma^3+\gamma^4+\gamma^5}$	α

The expected payoffs $v_{n,r}$ to the inspector

Table 4.

The reader will recognize that for $0 < r < n$, the payoffs are rational functions in γ and that the coefficients of the polynomials are arithmetic sequences of increasing orders, generated by the sequence 1, 1, 1, ...

In this section we obtain formulas for the expected payoffs and deduce from them some of the properties of the outcomes.

Theorem 5.1. The expected payoffs to the violator and to the inspector for the games $\Gamma(n, r)$, $0 < r < n$, when both players use optimal strategies are, respectively,

(5.1) $$w_{n,r} = \frac{P_{n-1,r}(\gamma)}{P_{n,r}(\gamma)} ,$$

(5.2) $$v_{n,r} = \frac{\alpha P_{n-1,r-1}(\gamma)}{P_{n,r}(\gamma)} ,$$

where

(5.3) $$P_{n,o}(\gamma) = 1 , \quad P_{n,n}(\gamma) = 1 , \quad n = 1, 2, \ldots ,$$

(5.4) $$P_{n,r}(\gamma) = \sum_{i=0}^{r} \binom{i+n-r-1}{i} \gamma^i , \quad n = 2, 3, \ldots , \quad 0 < r < n ,$$

and $\gamma = 1-\beta$.

The proof follows readily by induction, from the recursion formulas, and the cases $r = 1$ and $r = n-1$ are treated separately. It will be omitted.

Another expression for $P_{n,r}(\gamma)$, $0 \leq r < n$, is

(5.5) $$P_{n,r}(\gamma) = \frac{1}{(n-r-1)!} \frac{d^{n-r-1}}{d\gamma^{n-r-1}} P_{n,n-1}(\gamma) = \frac{1}{(n-r-1)!} \frac{d^{n-r-1}}{d\gamma^{n-r-1}} \frac{1-\gamma^n}{1-\gamma} .$$

Expressions of the type (5.4) appear in the study of the negative binomial distribution. (See, e. g., W. Feller [2].)

The above formulas enable us to deduce the following properties of the outcomes:

A. If γ is almost 0 , i. e., if a violation is relatively unimportant to the violator, then $w_{n,r}$ is near β (i. e., near 1) and $v_{n,r}$ is near α , $0 < r < n$.

B. If it is very important to the violator to break the treaty secretly; i. e., if γ is almost 1 , then

(5.6) $$w_{n,r} \sim \binom{n-1}{r} / \binom{n}{r} = 1 - r/n, \quad 0 \leq r \leq n, \quad n = 1,2,\ldots,$$

(5.7) $$v_{n,r} \sim \alpha \binom{n-1}{r-1} / \binom{n}{r} = \alpha r/n, \quad \binom{q}{-1} \equiv 0, \quad 0 \leq r \leq n, \quad n = 1,2,\ldots \quad .$$

In such a case, the inspector faces a situation almost equivalent to a situation in which the violator does not violate with a probability r/n , and violates being uninspected with a probability $1 - r/n$.

The violator can regard the situation as being almost equivalent to a situation in which he violates, and the probability that he is inspected is r/n .

Lemma 5.1. The expressions $P_{n-1, r}(t)/P_{n, r}(t)$ and $P_{n-1, r-1}(t)/P_{n, r}(t)$ are decreasing functions of t , $0 < r < n$, $0 \leq t < \infty$, $n = 2, 3, \ldots$.

The proof is straight forward, and it will be omitted.

C. It follows from Lemma 5.1, and (5.2), that if $0 < r < n$, then the more important it is for the violator to break the treaty secretly, the less is the expected payoff to both players[1].

6. Should the inspector announce his mixed strategy?

It is the purpose of this section to answer this question affirmatively.

Let us regard a pair of expected payoffs (x, y) , where x is the expected payoff to the inspector and y is the expected payoff to the violator, as a point in the euclidean plane; then if $0 < r < n$, all the

(1) It is assumed, of course, that both players know γ .

possible expected payoffs fill a closed triangle whose vertices are $(0, 1)$, $(1, 0)$ and (α, β) ; because these vertices are the only possible outcomes to the game(1).

The straight segment $[(0,1)\ (\alpha, \beta)]$ is, therefore, the Pareto optimum of the game; i. e., a point (x, y) belongs to this segment, if and only if any other point is less preferred by at least one of the players.

Theorem 6.1. The outcome $(v_{n,r}, w_{n,r})$ is Pareto optimal.

Proof. This is trivially true for $r = 0$ and for $r = n$. Let $0 < r < n$, $n = 2, 3, \ldots,$ then we have to show that

$$\frac{w_{n,r} - 1}{\beta - w_{n,r}} = \frac{v_{n,r}}{\alpha - v_{n,r}} \quad (6.1)$$

It follows from (5. 3) and (5. 4) that

$$P_{n,r}(\gamma) - P_{n-1,r}(\gamma) = \gamma P_{n-1,r-1}(\gamma) = (1-\beta)P_{n-1,r-1}(\gamma) ; \quad (6.2)$$

therefore,

$$P_{n,r}(\gamma) + \beta P_{n-1,r-1}(\gamma) = P_{n-1,r}(\gamma) + P_{n-1,r-1}(\gamma) . \quad (6.3)$$

(1) This triangle contains also points that can only be achieved if the players correlate their strategies - an unlikely situation, since there is no communication medium for such a correlation - the violator has signed a promise not to violate secretly and this rules out any agreement in which there is a promise to violate under certain conditions.

Multiplying each side of (6.3) by $P_{n,r}$ and subtracting the expression $P_{n-1,r}(\gamma)\,P_{n-1,r-1}(\gamma)$ from each side, we obtain

(6.4) $$[P_{n,r}(\gamma)]^2 + \beta P_{n-1,r-1}(\gamma)P_{n,r}(\gamma) - P_{n-1,r-1}(\gamma)P_{n-1,r}(\gamma) =$$
$$= P_{n,r}(\gamma)P_{n-1,r}(\gamma)+P_{n,r}(\gamma)P_{n-1,r-1}(\gamma) - P_{n-1,r}(\gamma)P_{n-1,r-1}(\gamma)\ .$$

By (5.1) and (5.2), this relation is equivalent to (6.1) . This completes the proof.

We can now claim that we have a case for recommending a disclosure of the inspector's mixed strategy, which is just as strong as the case for recommending a minimax strategy in a constant-sum game:

Indeed, by Theorem 4.1, the inspector can force an expected payoff arbitrarily near to $v_{n,r}$. By Theorem 6.1, the inspector would have achieved an expected payoff higher than $v_{n,r}$, only if the violator were to end up with an expected payoff which is less than $w_{n,r}$. But the violator can assure himself an expected payoff of at least $w_{n,r}$, by playing a minimax strategy based on his own payoff matrix regarded as a constant-sum game (Theorem 3.5). Thus, the only way for the inspector to gain more than $v_{n,r}$, is that the violator makes wrong assumptions and consequently ends up with an amount which is smaller than the amount he can guarantee for himself.

We ought to stress that this clear-cut answer could be achieved mainly because it is in the nature of this game that the inspector can announce his mixed strategy in a binding way, whereas the violator is not granted this privilege.

7. The case in which no player is allowed to announce his strategy.

Although, the above analysis yields a convincing answer to the particular game considered in this paper, one may wish to know what could happen if the inspector does not make use of his option; or, in general, what would the case be, if he cannot make the opponent believe that he will indeed use a certain mixed strategy. Then, we are back again to an analysis of a non-constant-sum game, with all its theoretical difficulties, and we have no definite answer to offer. However, we can, at least, speculate on some possibilities.

Conceivably, a player will assume that the other player is trying to harm him as much as possible, in which case he will choose a minimax strategy based on his own payoff matrix, regarded as a constant-sum game. We shall call such a strategy a pessimistic strategy. Another possibility is that a player will assume (or discover) that the other player is going to use a pessimistic approach, and he will therefore choose that (pure) strategy which will maximize his profits, if indeed his opponent will act as he is assumed to act. We shall call such a strategy - an optimistic strategy.

We shall omit the calculations - they are straight forward and employ Dresher's method [1], and shall merely state the results, assuming $0 < r < n$, since otherwise the game is determined.

(i) The inspector's pessimistic strategy $\{\tilde{q}_{n,r}(h(k))\}$ is

$$(7.1) \qquad \tilde{q}_{n,r}(h(k)) = \frac{\alpha P_{n^*-1, r^*-1}(\alpha)}{P_{n^*, r^*}(\alpha)} \qquad \text{if } 0 < r^* < n^* ,$$

$$(7.2) \qquad \tilde{q}_{n,r}(h(k)) = 0 \qquad \text{if } r^* = 0, \quad n^* > 0 ,$$

$$(7.3) \qquad \tilde{q}_{n,r}(h(k)) = 1 \qquad \text{if } r^* = n^* ,$$

where n^* and r^* are, respectively, the number of events and inspections which are left after history $h(k)$ has taken place.

If this strategy is chosen by the inspector, it will yield him an expected payoff equal to

$$(7.4) \qquad \tilde{v}_{n,r} = \frac{\alpha P_{n-1, r-1}(\alpha)}{P_{n,r}(\alpha)} ,$$

regardless of the strategy used by the violator.

It follows from Lemma 5.1, (5.2) and (2.1), that $\tilde{v}_{n,r} < v_{n,r}$, $0 < r < n$.

(ii) The violator's pessimistic strategy $\{\tilde{\ell}_{n,r}(h(k))\}$, where $\tilde{\ell}_{n,r}(h(k))$ is the probability to violate the event after the history $h(k)$, is

$$\text{(7.5)} \qquad \tilde{\ell}_{n,r}(h(k)) = 1 - \frac{w_{n^*, r^*}}{w_{n^*-1, r^*-1}} ,$$

if $0 < r^* < n^*$ and if no violation has occurred during the history $h(k)$,

$$\text{(7.6)} \qquad \tilde{\ell}_{n,r}(h(k)) = 1 ,$$

if $r^* = 0$, $n^* > 0$ and no violation has occurred dyring the history $h(k)$,

$$\text{(7.7)} \qquad \tilde{\ell}_{n,r}(h(k)) = 0 ,$$

if either $r^* = n^*$ or if a violation has occurred during the history $h(k)$. Here, n^* and r^* have the same meaning as in (i).

If this strategy is chosen by the violator, it will yield him an expected payoff equal to $w_{n,r}$, regardless of the strategy used by the inspector.

(iii) The inspector's optimistic strategy is to inspect the last r events.

(iv) <u>The violator's optimistic strategy is to violate only after all the available inspections were used up and if there are still events left.</u>

Consequently, the various payoffs can be deduced. They are summarized in Table 5.

		Violator: Pessimistic	Violator: Optimistic
Inspector	Pessimistic	$\tilde{v}_{n,r}$, $w_{n,r}$	$\tilde{v}_{n,r}$, $\hat{w}_{n,r}$
	Optimistic	$v_{n,r}$, $w_{n,r}$	α , β

Table 5

Here,

$$\hat{w}_{n,r} = \frac{\beta P_{n-1,r-1}(\alpha) + \binom{n-1}{r}\alpha^r}{P_{n,r}(\alpha)} , \tag{7.8}$$

and $\tilde{v}_{n,r}$ is defined in (7.4). Clearly, $\hat{w}_{n,r} > w_{n,r}$. Indeed, if the inspector takes his pessimistic strategy, and the violator takes his optimistic strategy, the violator acts as if he knew the inspector's strategy. The inspector's optimistic strategy differs from his optimal strategy described in Theorem 3.4, which is a unique minimax strategy based on the violator's payoff matrix. (See proof of Theorem 3.5).

Thus, the pessimistic strategy is not such a minimax strategy, and, therefore, the violator can assure himself more than $w_{n,r}$ if he acts properly.

We see from Table 5, that the only way (among these alternatives) for the inspector to get more than $v_{n,r}$ is if

(a) each player wrongly assumes that his opponent is taking a pessimistic strategy,

(b) each player takes the risk and adopts the optimistic strategy.

8. Interpersonal comparisons of utilities

We have recommended that the inspector announces a strategy which almost opposes the interests of the violator. It is chosen in such a way as to give the violator a bonus of "ε", if and only if he conforms to the wait-until-all-the-available-inspections-were-used-up strategy.

It may be argued that such a small prize may not satisfy the violator. He would rather give up the ε, and cause the inspector a heavy loss.

We claim that this argument is not valid in our game! If the violator is rational, as we assume he is - why would he sacrifice his ε to begin with?

The problem <u>could</u> arise if the violator was in a position to make a threat of the form: "If you do not give me a substantial bonus, I shall act differently, whereby I shall suffer only a small loss, but your damage will be great". Such a threat, if it were possible, would have brought us to questions concerning interpersonal comparisons of utilities, to which game theory does not at present have a satisfactory answer(1). Fortunately, in our game, the violator cannot make such a threat; i.e., he cannot announce that he will violate under certain circumstances, because he has signed a promise never to violate, and his signature blocks his communication medium.

9. Some practical aspects.

It turns out that the optimal strategy for the inspector depends only on the violator's payoff matrix; namely on β. In real life situations, however, it is extremely hard to expect that the precise value of β is known to the inspector. Aside from objective aspects of the situation, which, perhaps, can be estimated, the violator may have

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(1) But it is, perhaps, safe to say that the players would settle somewhere in the intervals $([v_{n,r}, v_{n,r}], [w_{n,r}, \hat{w}_{n,r}])$.

antagonistic feelings toward the inspector, which enter into the value of β . These are next to impossible to measure, because the inspector cannot "interview" the violator on such aspects and expect an honest answer. Moreover, these antagonistic feelings may well depend on the actions taken by the inspector, for each inspection may mean an insult.

We do not see how to overcome these difficulties within the framework of a mathematical theory. Qualitatively speaking, however, if the antagonistic feelings are not too great, the inspector should give the violator a substantial bonus, to cover up for the inaccuracy of estimating β . He would achieve this if he underestimates β (i.e., overestimates γ), as our analysis in Sections 3 and 4 shows.

Note that even if the inspector places $\beta \approx 0$ (i.e., $\gamma \approx 1$), his expected payoff (5.7) will be greater than the amount $\tilde{v}_{n,r}$ he can guarantee for himself by playing a pessimistic strategy. This follows from Lemma 5.1, (2.1), (5.2) and (7.4).

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