A GENERAL THEORY OF MEASUREMENT
APPLICATIONS TO UTILITY*

Johann Pfanzagl†

For a long time measurement was confined to sciences such as geometry, astronomy, and physics. It is, therefore, quite natural that the theory of measurement was restricted to the special circumstances usually encountered in these sciences, to additive magnitudes such as "mass" or "electrical resistance" or "length." In psycho-physics, psychology, or welfare economics, on the other hand, additive magnitudes are hardly present. Therefore, new methods had to be developed—in psycho-physics, for example, the method of "bisection," successfully applied by S. S. Stevens and others (e.g., [16, 17]) in obtaining scales for subjective magnitudes such as pitch, loudness, etc., and in econometrics the method of Morgenstern-von Neumann for measuring subjective utility.

Because these methods are not covered by the traditional theory of measurement, it was argued (especially in discussions concerning measurement in psycho-physics and psychology, e.g., [3]), that they do not lead to genuine measurement but only to something in the nature of an ordinal scale. In the field of psychology, especially, this criticism was supported by the fact that there was no adequate theory as well founded as the theory of measuring additive magnitudes. The situation concerning the measurement of utility is quite the opposite of course: Here, we have an incontestable theory, developed by Morgenstern and von Neumann [11], and it is rather the empirical investigations which are lacking.

In the following sections we give a brief outline of a general theory of measurement and then its applications to the measurement of utility.

OUTLINE OF A GENERAL THEORY OF MEASUREMENT

The general aim of "measurement" is to map a set M on the set of real numbers in such a way, that—to the greatest possible extent—conclusions concerning the relations between elements of M can be drawn from corresponding relations between their assigned numbers. A trivial example: If a set is ordered, the mapping is performed in such a way that the order-relation between the elements of M is reflected by the order-relation of their assigned numbers, or if there is an additive operation defined between each pair of elements of M, the mapping is performed in such a way, that the assigned number of a sum of elements in M equals the sum of the assigned numbers of these elements.

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We now consider an ordered set M, for which there is defined an operation, called a "metric" operation, which fulfills the following set of axioms:

1. Existence Axiom: For each pair of elements $a, b \in M$ there exists a unique element $a \circ b \in M$.

2. Monotonicity Axiom: If $a \lesssim a'$, then $a \circ b \lesssim a' \circ b$ for all $b \in M$.

   If $b \lesssim b'$, then $a \circ b \lesssim a \circ b'$ for all $a \in M$.

3. Continuity Axiom:[1] The operation $a \circ b$ is continuous for both a and b.

4. Bisymmetry Axiom: $(a \circ b) \circ (c \circ d) \sim (a \circ c) \circ (b \circ d)$.

THEOREM 1. An ordered, connected set M, for which a metric operation is defined, can be mapped into the set R of real numbers ($a \in M$, $a \to a^* \in R$) in such a way that:

1. The mapping $a \to a^*$ is continuous,
2. The mapping $a \to a^*$ is monotone, i.e., $a \lesssim b$ implies $a^* \leqq b^*$,
3. The operation "o" is mapped isomorphically on a linear operation: $(a \circ b)^* = pa^* + qb^* + r$.

Mappings of this kind are unique up to linear transformations. An exact proof of Theorem 1 is given in [13, pp. 49-51]. If one does not insist on ultimate mathematical precision, one can assume in advance the possibility of a preliminary mapping $a \to a' \in R$ which is continuous and monotone. Then, by $(a \circ b)' = F(a', b')$ a function is defined which is - due to the metric axioms - unique, continuous, monotone and bisymmetric:

$$F[F(a',b'), F(c',d')] = F[F(a',c'), F(b',d')] .$$

In [1] Aczél has proved that functions of this kind can be expressed by a function $f(x')$ in the following way:

$$F(a',b') = f^{-1}[pf(a') + qf(b') + r] .$$

If we put $x^* = f(x')$ it follows immediately, that:

$$(a \circ b)^* = f[F(a',b')] = pf(a') + qf(b') + r = pa^* + qb^* + r .$$

As the scale is unique up to linear transformations, the magnitudes p and q are uniquely determined by the operation "o." They are invariant under any linear transformation. (According to the monotonicity of the metric operation both p and q have to be positive.) In the general case $p + q \neq 1$, r is not invariant under linear transformations. Therefore, if the origin of the scale is chosen in an appropriate manner, r vanishes, so that $(x \circ y)^* = px^* + qy^*$.

[1] For a precise mathematical statement of this Axiom see [13, p. 20].

Hence, in the regular case ($p + q \neq 1$), there exists a natural origin, and the scale is unique up to multiplication by a constant.

In the singular case $p + q = 1$, even the constant r is invariant under linear transformations and therefore uniquely determined by the metric operation "o." Hence, no natural origin is fixed by the metric operation itself in the singular case: additional aspects have to be taken into account in order to fix the origin and to achieve a scale unique up to linear transformations. (For instance, if M has a smallest element, it is quite natural to require that this smallest element shall be assigned the number 0.)

A very important example of the singular case is the reflexive operation, for which $aoa \sim a$ for each $a \in M$. The fact that $p + q = 1$ in this case follows immediately from relation

$$a^* = (aoa)^* = pa^* + qa^* + r = (p + q)a^* + r .$$

Furthermore, we conclude from this relation that $r = 0$ for all reflexive operations.

The question as to under which circumstances two different metric operations lead to the same scale is answered by the following, Theorem 2:

THEOREM 2: Two different metric operations "o" and "•" lead to scales identical up to linear transformations, if for each quadruple a, b, c, $d \in M$ the following isometry-relation holds: $(aob) \bullet (cod) \sim (a \bullet c)\, o\, (b \bullet d)$.

For a proof of Theorem 2 see [13, p. 24]. The precise meaning of Theorem 2 is the following. Each of the operations "o" and "•" leads to a scale, which will be designated by $a \to a^*$ and $a \to a^{**}$ respectively:

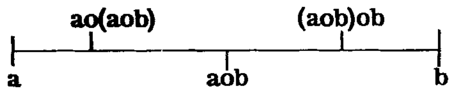
$$(aob)^* = p_o a^* + q_o b^* + r_o ,$$

$$(a \bullet b)^{**} = p_1 a^{**} + q_1 b^{**} + r_1 .$$

Theorem 2 states that the two scales a^* and a^{**} differ only by a linear transformation if the two operations are isometric. The parameters $p_o, p_1;\ q_o, q_1;$ and r_o, r_1 will, in general, be different, of course. If at least one of the two operations is non-singular (e.g., $p_o + q_o \neq 1$), we can choose a scale such that $r_o = 0$: $(aob)^* = p_o a^* + q_o b^*$. If the two operations are isometric, then also $r_1 = 0$: $(a \bullet b)^{**} = p_1 a^{**} + q_1 b^{**}$. This means that the natural origins are identical for both scales.

The general theory outlined above covers, of course, the traditional case of an additive magnitude. From associativity and commutativity, the bisymmetry follows immediately, so that each additive operation is a metric operation in the sense defined above. Therefore, according to Theorem 1, a continuous and monotone mapping exists, such that $(aob)^* = pa^* + qb^* + r$. Furthermore, as the additive operation is associative and commutative, $p = q = 1$ holds. Therefore, the origin of the scale can be chosen such that $r = 0$, so that we finally obtain a scale, for which $(aob)^* = a^* + b^*$ which corresponds to the traditional result.

Also bisection, widely used by S. S. Stevens and others (e.g., [16, 17]) to construct subjective scales of loudness, pitch, etc., seems to be a metric operation. The only relation whose validity could be questioned is the bisymmetry. The approach in the experiments was a purely empirical one, and there was no precise knowledge as to what conditions the operation of bisection had to fulfill in order to allow for the construction of a cardinal scale. Therefore, no information concerning the validity of the bisymmetry axiom exists. But there is at least an indirect indication that the bisymmetry axiom might be valid. In the case of bisection,



aob is the magnitude midway between a and b, ao(aob) the magnitude midway between a and (aob), (aob)ob the magnitude midway between (aob) and b. As the bisection is also commutative and reflexive, it follows, from bisymmetry, that (aob) should be midway between ao(aob) and (aob)ob. According to the experiments performed, this relation seems to hold at least for the measurement of pitch. This is a hint that the bisymmetry axiom holds in the case of pitch and that bisection is therefore a metric operation.[2] In this case, according to our Theorem 1, a mapping exists, for which $(aob)^* = pq^* + qb^* + r$. As stated above, the bisection is reflexive. It is therefore a singular metric operation, and no natural origin is determined by the operation itself. The origin can, nevertheless, be assigned in quite a natural way by assigning the number zero to the most extreme sensation at the lower bound (the least degree of loudness, the deepest pitch, etc.). However, the origin might be fixed arbitrarily; since the operation is singular, this has no influence over r: According to the assumption of reflexivity, $r = 0$ in any case. Together with commutativity, reflexivity yields that $p = q = \frac{1}{2}$. Hence, there exists a scale, unique up to linear transformations, such that

$$(aob)^* = \frac{1}{2}(a^* + b^*) .$$

This is exactly the result Stevens assumed to hold.

An additional example of the application of the general theory of measurement is the measurement of utility, which will be treated in detail in the next section.

THE MEASUREMENT OF SUBJECTIVE UTILITY ACCORDING TO MORGENSTERN AND von NEUMANN

In the following remarks it will be shown that the general theory outlined above can be applied to the measurement of subjective utility according to the concept of Morgenstern and von Neumann.

M is interpreted as the set of the situations to be valued. With Morgenstern and von Neumann we assume that a complete order is defined for the elements of M; (see [11, p. 26, axiom 3A]).

In [13, pp. 53, 54] it is shown that an ordered set M which fulfills the axioms of Morgenstern and von Neumann is connected and that the operation "$a\alpha b$" fulfills for each $\alpha \epsilon (0,1)$ the metric axioms. Therefore, the operation signified by α can be regarded as a special case of the metric operation "o." Furthermore, our isometry-relation $(a\alpha b)\, \beta (c\alpha d) \sim (a\beta c)\, \alpha (b\beta d)$ is fulfilled for any $\alpha, \beta \,\epsilon (0, 1)$. This has the following consequences:

[2]On the other hand, experiments reported by Gage [6] suggested that the relation may not hold for loudness. See also [12].

1. For the construction of a cardinal scale it is sufficient to consider alternatives with any fixed probability α which can be chosen at will. It is not necessary to use alternatives with differing probabilities. For the special case of alternatives with the subjective probability $\frac{1}{2}$ this has already been pointed out in 1926 by Ramsey [14]. As the operation "$a\alpha b$" is reflexive, we get a cardinal scale of utility $U(x)$, for which $U(a\alpha b) = pU(a) + (1-p)U(b)$. This scale is unique up to linear transformations.

2. As the isometry-relation is fulfilled we get, on the basis of alternatives with a probability $\beta \neq \alpha$, identical scales (i.e., scales differing only by a linear transformation) of utility $U(x)$, but different weights p', $(1-p')$. Therefore, $U(a\beta b) = p'U(a) + (1-p')U(b)$.

Our assumptions suggest that the weights p and p' should be interpreted as subjective probabilities which are assigned to the objective probabilities α and β; respectively, $p = s(\alpha)$, $p' = s(\beta)$. The assumption that the operation $a\alpha b$ fulfills not only the metric axioms but all axioms of Morgenstern and von Neumann, implies the identity of the subjective probability $s(\alpha)$ with the objective probability α. Axiom 3:C:b [11, p. 26] states that $(a\alpha b)\beta b \sim a\alpha\beta b$. This implies $s(\alpha\beta) = s(\alpha)\cdot s(\beta)$. Together with $s(1-\alpha) = 1-s(\alpha)$, this leads to the result: $s(\alpha) = \alpha$.

However, it is not absolutely necessary to suppose that the axioms of Morgenstern and von Neumann are fulfilled. If our (weaker) metric axioms are fulfilled, this is sufficient for the construction of a cardinal scale of utility of the following form: $U(a\alpha b) = s(\alpha)U(a) + (1-s(\alpha))U(b)$. If, in addition, the isometry-relation is also fulfilled, then the same scale of utility is obtained, whatever probability is taken as a base for the alternative $a\alpha b$.

These assumptions are weaker than those of the system of axioms of Morgenstern and von Neumann. In spite of that, they permit us to derive all relevant results concerning the scale of utility, admitting, however, a divergence between subjective and objective probability.

It should be noted that, on the basis of the general theory of measurement, the subjective probability assigned to α arises quite naturally out of the procedure of measurement. It is not necessary to anticipate its existence and other properties in a separate system of axioms in addition to the axioms concerning the operation $a\alpha b$. It would be worthwhile to undertake a critical examination of the consequences resulting from this fact for the theories which—following the example of Ramsey—start from a common axiom system for utility and subjective probability, such as those of Savage [15] and Luce [8].

Until now, the validity of the metric axioms and of the isometry-relation is derived from the axioms of Morgenstern and von Neumann. We will now try to check the evidence of the metric axioms and of the isometry-relation directly. We shall restrict this undertaking to measuring subjective utility with regard to different quantities of an identical commodity, e.g., of money. By this restriction the problem can be treated more precisely and the analysis of the necessary assumptions can be carried through more thoroughly.

In the subsequent analysis the elements $x \in M$ therefore stand for different quantities of the same commodity. Furthermore, an event is given, the occurrence of which is uncertain. Let us designate the occurrence of this event by P, its non-occurrence by $\overline{P}$. The wager "aPb" with $a,b \in M$ means: The individual in question gets quantity a, if P occurs, quantity b, if $\overline{P}$ occurs. Let $\{P,\overline{P}\}$ be an event which can be repeated indefinitely, e.g., a random experiment. (In principle the repetition could also consist in the fact that a unique event $\{P,\overline{P}\}$ is judged by different persons with identical preference scales.) Let $m(aPb)$ be the quantity which has the same subjective utility as the wager "aPb". As is shown even by every-day experience, the quantity $m(aPb)$ is not uniquely determined without further conventions.

In order to determine this quantity uniquely the following method is used (see [10]). Let w(x,y) be the probability that the individual prefers x, if faced with the alternatives x and y. m(aPb) is then defined by $w(m(aPb), aPb) = \frac{1}{2}$. Of course, any deviation of m(aPb) from the objective expected value (in the sense of the probability calculus) of the wager aPb is possible.

After these preliminary remarks we shall now proceed to an examination of the evidence of our metric axioms.

1. Existence axiom: To the two elements $a,b \in M$ we assign the element $m(aPb) \in M$. This merely assumes that the above outlined definition of m(aPb) is meaningful.

2. Monotonicity axiom: The validity of the monotonicity axiom seems to be evident since we have confined the definition of m(aPb) to a,b being different quantities of the same commodity.

3. Continuity axiom: Also this axiom seems evident in consequence of the restriction to different quantities of the same commodity.

4. Bisymmetry axiom: $m[m(aPb)\,Pm(cPd)] = m[m(aPc)\,Pm(bPd)]$. Let P and Q be two different events. Then certainly $(aPb)\,Q(cPd) \equiv (aQc)\,P(bQd)$. For, whichever of the combinations $PQ, \overline{P}Q, P\overline{Q}, \overline{P}\overline{Q}$ is realized, the result will always be the same for both $(aPb)\,Q(cPd)$ and $(aQc)\,P(bQd)$, namely: $PQ \rightarrow a$, $\overline{P}Q \rightarrow b$, $P\overline{Q} \rightarrow c$, $\overline{P}\overline{Q} \rightarrow d$. In view of this fact, we have used the symbol of identity "≡" in order to distinguish this relation from the equivalence "~" defined in terms of utility. The decisive assumption is that $uPv \sim m(uPv)$ and $xPy \sim m(xPy)$ implies $(uPv)\,Q(xPy) \sim m(uPv)\,Qm(xPy)$. By means of this relation we can deduce from the identity stated above that $m(aPb)\,Qm(cPd) \sim m(aQc)\,Pm(bQd)$. Furthermore, we must assume that $uPv \sim xQy$ (i.e., $w(uPv, xQy) = \frac{1}{2}$) implies $m(uPv) = m(xQy)$. Then we get $m[m(aPb)\,Q\,m(cPd)] = m[m(aQc)\,Pm(bQd)]$. This is the isometry-relation which guarantees that the events $\{P, \overline{P}\}$ and $\{Q, \overline{Q}\}$ lead to the same scale of utility. To obtain the bisymmetry axiom we need only assume that $\{Q, \overline{Q}\}$ is a repetition of the experiment $\{P, \overline{P}\}$, independent of the result which this experiment has yielded. Then $m(xQy) = m(xPy)$, and thus $m[m(aPb)\,Pm(cPd)] = m[m(aPc)\,Pm(bPd)]$.

5. Reflexivity axiom: $m(aPa) = a$ seems evident, as the wager aPa leads with certainty to the result a.

The relation $m(aPb) = m(b\overline{P}a)$ does not involve an additional assumption; it holds a priori as a consequence of the designation we are using here.

Summing up, we can say that the validity of the metric-axioms, plausible as they are, still needs a critical empirical verification. The possibility does not seem to be excluded that e.g., experiments with wagers xP*y with a subjective probability $\frac{1}{2}$ will show that $m(xP^*y) = p\,\mathrm{Max}(x,y) + (1-p)\,\mathrm{Min}(x,y)$; p being any number between 0 and 1. For $p \neq \frac{1}{2}$ the bisymmetry axiom is not fulfilled; in this case, the construction of a metric-scale of utility on the basis of the wagers xP*y would therefore be impossible. In spite of that, the behavior of the person in question could not be called irrational.

THE CONSISTENCY AXIOM

The following section considers the meaning and the implications of an additional assumption concerning the evaluation of wagers

Consistency axiom: $m[(a + c)\,P(b + c)] = m(aPb) + c.$

The axioms dealt with above were exclusively concerned with the evaluation of alternative events. In the case of the consistency axiom the evaluation of conjunctive events is considered. We designate the conjunctive connection of two events x,y by "x&y" (both x and y). Then certainly $(a + c) P(b + c) \equiv (aPb) \& c$. The right-hand side of the equation means: the amount c is paid in any case, and, in addition, the amount a is paid in the case of P, and the amount b in the case of $\bar{P}$. This is identical with paying the amount $(a + c)$ in the case of P, the amount $(b + c)$ in the case of $\bar{P}$. If one assumes that $aPb \sim m(aPb)$ implies $(aPb) \& c \sim m(aPb) \& c$, then one gets: $(a + c) P(b + c) \sim m(aPc) \& c \equiv m(aPc) + c$. This means by definition: $m[(a + c) P(b + c)] = m(aPc) + c$, as stated by the consistency axiom.

Though the consistency axiom was up to now not stated explicitly anywhere, it can be shown that several authors are really tacitly assuming it. In the following, we will show this has been so in the case of Friedman and Savage [5], Mosteller and Nogee [10], and von Neumann and Morgenstern [11].

Using the designation adopted in this paper,[3] a statement by Friedman and Savage in [5, p. 290] reads: "If $m(a'\alpha b')$ is greater than x, the consumer unit (purchaser) prefers this particular risk (namely, the participation in the lottery $a'\alpha b'$) to a certain income of the same actuarial value and would be willing to pay a maximum of $[m(a'\alpha b') - x]$ for the privilege of 'gambling'." This statement by Friedman and Savage is confined to the case where x is the actuarial value of the lottery $(a'\alpha b')$. But this argument, if valid for the actuarial value, is obviously valid for any value x, so that we get the equivalent statement: "If $m(a'\alpha b')$ is greater than x, the purchaser prefers this particular risk (namely, the participation in the lottery $a'\alpha b'$) to a certain income of the value x and would be willing to pay a maximum of $[m(a'\alpha b') - x]$ for the privilege of 'gambling'."

If the purchaser pays an amount $[m(a'\alpha b') - x]$ for the lottery ticket $(a'\alpha b')$, he actually faces the chances of getting the amount $[a' - m(a'\alpha b') + x]$ with probability α and $[b' - m(a'\alpha b') + x]$ with probability $(1-\alpha)$. The utility of this risk is—according to the quotation above—equal to the utility of the amount x: The purchaser is willing to pay $[m(a'\alpha b') - x]$ for the lottery ticket; therefore: $U([a' - m(a'\alpha b') + x]\ \alpha\ [b' - m(a'\alpha b') + x]) \geqq U(x)$. But the purchaser is not willing to pay more than $[m(a'\alpha b') - x]$. Therefore: $U([a' - m(a'\alpha b') + x]\ \alpha\ [b' - m(a'\alpha b') + x]) \leqq U(x)$. It follows: $U([a' - m(a'\alpha b') + x]\ \alpha\ [b' - m(a'\alpha b') + x]) = U(x)$. This means: $x = m([a' - m(a'\alpha b') + x]\ \alpha\ [b' - m(a'\alpha b') + x])$. By putting $m(a'\alpha b') - x = c$ we obtain: $m(a'\alpha b') - c = m[(a'-c)\ \alpha\ (b'-c)]$. If we put further $a' = a + c$, $b' = b + c$, we obtain: $m[(a + c)\ \alpha\ (b + c)] = m(a\alpha b) + c$. Therefore the validity of the argument used by Friedman and Savage implies the validity of the consistency axiom. (As a and b are incomes in the case discussed by Friedman and Savage, we have assumed in the derivation given above, that there is no other amount of money to be taken into account. But this was only for sake of brevity. The same argument holds if one regards $s + a' - m(a'\alpha b') + x$, $s + b'm(a'\alpha b') + x$ and $s + x$ instead of $a' - m(a'\alpha b') + x$, $b' - m(a'\alpha b') + x$ and x respectively.

Mosteller and Nogee [10, p. 399], in discussing the results and conditions of their experiment, consider also the "effect of the amount of money in front of the subject upon his decisions" to take part in the gamble or not. One of their statements is this: "One possible criticism could be that the subject changes his utility curve with these changes in capital, so that each decision he makes depends on the amount of money he has on hand at that particular moment." It seems remarkable that Mosteller and Nogee are thinking only of the possibility

[3] The translation in the designation is: $I_1 \to a'$, $I_2 \to b'$, $I^* \to m(a'\alpha b')$, $I \to x$.

that the available amount of money could change the utility function of the player. They do not consider the possibility that the willingness to accept a particular game could depend on the amount of money held by the individual, even if his utility function were unchanged. And they are only thinking of "the amount of money in front of the subject"—not of the amount in his pocket. This becomes especially clear when they are talking (in a different connection, p. 403) of one of the participants, who became unemployed during the course of the study. It appears that these authors had neglected the possibility that this might have changed his behavior concerning the participation in these games. If this were true, they would have assumed that the utility of a special game is independent of the money held by the individual (perhaps with an exception regarding the money immediately involved in playing). This assumption, however, is equivalent to the consistency axiom.

The decisive question seems to be this: \$10 α \$1 is a lottery, offering an amount of \$10 with probability α and an amount of \$1 with probability $1-\alpha$. If an individual is willing to pay \$2 for a ticket of this lottery (and not more than \$2), this does not necessarily mean that, for this individual, \$10 α \$1 is equivalent to \$2, i.e., $m(10\ \alpha\ 1) = 2$. It actually means that the status quo has the same utility as a lottery, which leads to the status "quo + \$8" with probability α and status "quo - \$1" with probability $1-\alpha$. (We must insert \$8 and - \$1 as net outcomes of the lottery, since we have to deduct the \$2 spent for the lottery ticket from the gross prices of \$10 and \$1, respectively.) If we assume that for this purpose the status quo can be described essentially by stating the amount of money s held by the individual, then the willingness to pay at most \$2 for the lottery ticket really means that $m[(s + 8)\ \alpha\ (s-1)] = s$, not $m(10\ \alpha\ 1) = 2$, as stated above. Both statements are equivalent only if the consistency axiom holds.

It would require a thorough investigation as to what the real meaning of experiments like those performed by Mosteller and Nogee [10], or by the Applied Mathematics and Statistics Laboratory of Stanford University (e.g., [4] or [18]), could be, if the consistency axiom were not fulfilled.

Finally, the consistency axiom is also used in the theory of games. A good example is von Neumann-Morgenstern's introduction of the concept of "strategic equivalence" in [11, p. 245 ff]. My attention was drawn by Oskar Morgenstern and Harlan D. Mills to the results obtained by Kemeny, DeLeeuw, Snell, and Thompson [7]. According to a quotation in [9, p. 72], in examining the restrictions imposed on the utility function by the concept of strategic equivalence these authors have shown in [7] that the utility function-assuming strict monotonicity and differentiability must be one of the types stated in Theorem 3.

THEOREM 3. If utility is measurable by a cardinal scale such that $U(m(aPb)) = pU(a) + (1-p)U(b)$, if furthermore the consistency axiom holds, and if the utility function is assumed to be continuous and strictly monotone, then the function is of one of the following types:

$$U_0(x) = Ax + B \quad A > 0 ,$$

$$U_1(x) = A\lambda^x + B \quad A > 0, \lambda > 1 \text{ or } A < 0, 0 < \lambda < 1 .$$

If U(x) is standardized such that U(0) = 0,U(1) = 1, we obtain:

$$U_o(x) = x$$

$$U_1(x) = \frac{1-\lambda^x}{1-\lambda}.$$

In $U_1(x)$ we must distinguish between the case $\lambda > 1$ and $0 < \lambda < 1$. In both cases $U_1(x)$ is monotone increasing; however, the marginal utility U'(x) decreases only in the case $0 < \lambda < 1$ with increasing x. In this case

$$\lim_{x \to \infty} U_1(x) = \frac{1}{1-\lambda}.$$

These statements concerning the shape of the scale of utility cannot be verified immediately. They can, however, be put in a way which allows an experimental proof or disproof. The whole theory of the measurement of utility depends on the fact that the magnitude m(aPb) can be clearly determined by means of experiments. This, however, permits us to transform the statements concerning the shape of the scale of utility into statements concerning directly observable magnitudes: It follows from the two solutions $U_o(x)$, $U_1(x)$, that m(aPb) must be a function of one of the following types:

$$m_o(aPb) = pa + (1-p)b$$

$$m_1(aPb) = \log_\lambda [p\lambda^a + (1-p)\lambda^b].$$

Both functions fulfill, as can be easily verified, the metric-axioms and the consistency axiom. Furthermore the isometry-relation is fulfilled if λ is independent of the special event $\{P,\bar{P}\}$. Given the subjective probability p, the behavior of the individual can be described by only one parameter, λ.

Proof of theorem 3: From U(m(xPy)) = pU(x) + (1-p)U(y) and m[(x + z)P(y + z)] = m(xPy) + z together we obtain the functional equation:

$$pU(x+z) + (1-p)U(y+z) = U\{U^{-1}[pU(x) + (1-p)U(y)] + z\}, \quad 0 < p < 1.$$

We are considering continuous and strictly monotone solutions of this functional equation.

Let $y = 0$: $pU(x+z) + (1-p)U(z) = U\{U^{-1}[pU(x) + (1-p)U(0)] + z\}$

Let $x = 0$: $pU(z) + (1-p)U(y+z) = U\{U^{-1}[pU(0) + (1-p)U(y)] + z\}$.

By adding these two equations and using the functional equation stated above again, we obtain:

(1)
$$U\{U^{-1}[pU(x) + (1-p)U(y)] + z\} + U(z) =$$
$$= U\{U^{-1}[pU(x) + (1-p)U(0)] + z\} + U\{U^{-1}[pU(0) + (1-p)U(y)] + z\}.$$

If we introduce the following:

$$p[U(x) - U(0)] = \xi$$

$$(1-p)\,[U(y) - U(0)] = \eta$$

$$U\{U^{-1}[\zeta + U(0)] + z\} - U(z) = f(\zeta, z)\ ,$$

then it follows from (1) that

$$f(\xi + \eta, z) = f(\xi, z) + f(\eta, z)\ .$$

$f(\zeta, z)$ is for each z a continuous and strictly monotone function of ζ.

Hence: $f(\zeta, z) = \zeta\ \varphi(z)$, i.e., $U\{U^{-1}[\zeta + U(0)] + z\} - U(z) = \zeta\ \varphi(z)$. We designate: $U(x) - U(0) = \Psi(x)$ and obtain: $\Psi\{\Psi^{-1}(\zeta) + z\} = \zeta\ \varphi(z) + \Psi(z)$. For $\zeta = \Psi(x)$ it follows:

$$\Psi(x + z) = \Psi(x)\ \varphi(z) + \Psi(z)\ . \tag{2}$$

As $\Psi(x + z) = \Psi(z + x)$, we obtain:

$$\Psi(x)\ \varphi(z) + \Psi(z) = \Psi(z)\ \varphi(x) + \Psi(x)\ .$$

As $U(x)$ is strictly monotone, $\Psi(x) \neq 0$ for $x \neq 0$, hence:

$$\frac{\varphi(z) - 1}{\Psi(z)} = \frac{\varphi(x) - 1}{\Psi(x)}\ ,\ \text{i.e.,}\ \frac{\varphi(x) - 1}{\Psi(x)} = c\ .$$

This result introduced in (2) gives:

$$\Psi(x + z) = c\,\Psi(x)\,\Psi(z) + \Psi(x) + \Psi(z)\ .$$

This is a functional equation of the type $\Psi(x + z) = F[\Psi(x), \Psi(z)]$. The continuous and strictly monotone solutions of such a functional equation are essentially unique: If $\Psi_0(x)$ is a continuous and strictly monotone solution, then all functions $\Psi_0(ax)$ are solutions and only these (cf. Aczél [2, p. 120, 121]). We have to distinguish between two cases:

$c = 0$: general solution: $\Psi_0(x) = ax$

$c \neq 0$: general solution: $\Psi_1(x) = \frac{1}{c}(\lambda^x - 1)$.

From $\Psi(x) = U(x) - U(0)$ we obtain the theorem to be proved.

CONCLUSIONS

The applications of the results of a general theory of measurement, developed in [13], to problems of utility show that the valuation of wagers with a constant probability is sufficient for the construction of a cardinal scale. This cardinal scale allows for a divergence between subjective and objective probability. Even this "weak" cardinal scale, together with the consistency axiom introduced in this paper, restricts the utility functions essentially to a one-parameter family. If, on the other hand, the consistency axiom is not valid, a more thorough investigation would be needed in order to determine the real meaning of the utility functions obtained by experiments with gambling.

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