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On the Definition of Differentiated Products in the Real World*

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Summary. This paper proposes an abstract model of commodity differentiation that incorporates manufacturing imprecision and dimensioning and tolerancing standards. The potential consistency of such a model based on engineering consideration is analyzed. For a large pure exchange economy, competitive equilibria exist and are Pareto optimal. Production issues such as the derived demand for intermediate products, continuity of cost functions, and product selection and technology issues such as mass customization, agile manufacturing, and manufacturability are discussed.

Key words: Differentiated commodities, General equilibrium, Hausdorff metric topology.

JEL Classification Numbers: D51, L15, D21.

2.1 Introduction

This paper proposes a new way to formulate commodity spaces in microeconomic theory that is both more specific and more abstract than standard definitions of commodity spaces, including those for differentiated commodities, in the existing literature. I focus on uncertainties inherent in any production technology and aim for consistency with how commodities are actually purchased. The overall goal is to demonstrate that one can modify our standard model in microeconomic theory so that it reflects these concerns yet nevertheless remains tractable for economic analysis. Then the resulting economic properties can be examined and compared to those of the existing benchmark model of an economy.

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This research is motivated by theoretical models of engineering design and manufacture, especially the solid geometric modelling work underpinning computer aided design (CAD) and computer assisted manufacturing (CAM) tools. For concreteness and simplicity, I have chosen to focus on geometric forms such as precision metal parts and dies for plastic molding. This offers the advantage of easy visualization, but note that the same engineering principles would carry over to other types of commodities.

An important aspect of any manufacturing procedure is its level of precision – the closeness of the actual manufactured object to the desired object that is specified in the design and the reproducibility of the operation with the process remaining under control without further interference. The subfield of dimensioning and tolerancing (D & T) studies this uncertainty, how it can and should be measured, how it is modelled formally, how its specifications should be standardized (i.e., ANSI 14.5 in the U.S. and ISO 9000 internationally), and how a given level of uncertainty affects production costs and possible time-to-market delays in the introduction of new products.

Yet, for economics, it is essential that any useful formal model be analytically tractable and display the potential to yield interesting economic conclusions. Thus there must be a balance between increased generality and abstraction on the one hand and the prospects for obtaining interpretable economic results on the other hand. One fruitful approach is to delineate clearly the comparisons and contrasts between a benchmark model and the proposed novel approach, while a related research strategy consists of displaying exactly the sense in which one model encompasses the other. This analysis is performed here for my proposed model versus Mas-Colell's (1975) renowned model of abstract commodity differentiation with indivisibilities. As a bonus, the presence of indivisibilities in the differentiated commodities (geometric objects) here is natural and intuitive.

In economic theory, Debreu (1959) pointed out the necessity of formalizing the definition of the set of commodities present in an economy. His well-known, well-exposited, and well-reasoned statement on this matter appears as Chapter 2. There he argues that a commodity should be described in terms of its complete physical description, its location, and its date of delivery so that all units of a given single commodity would be viewed as completely equivalent by each consumer and each firm in the economy. This paper focuses on the physical description aspect of the definition of a commodity and suggests that how economists think about physical descriptions of goods can be improved. My proposed improvement is consistent with actual (incomplete) contracts to purchase and sell goods – for instance, defense procurement – and features contracts that are, in principle, legally enforceable as the basis for defining commodities. In addition, my framework respects realistic limits on information with respect to the physical characteristics of products in that economic agents are not hypothesized to take account of nonverifiable information about the production process.

Debreu's (1959) admonishment to pay careful attention to the specification of the commodity space in economic theory has been followed up by a long list of researchers – e.g., Bewley (1972) and many others who have examined various infinite-dimensional commodity spaces in general equilibrium theory and Prescott and Townsend (1984), who advocate randomizations as a convexification device (later utilized for different purposes by Hornstein and Prescott, 1991, and by Cole and Prescott, 1995). My paper builds on the seminal article by Mas-Colell (1975), which provides a state-of-the-art model of abstract commodity differentiation.

However, to incorporate engineering considerations of product design and manufacture, it is necessary to add several layers to the Mas-Colell (1975) approach so that it reflects the specific structure of the commodity space suggested by geometric design theory and by dimensioning and tolerance analysis. This involves much more than simply adding uncertainty or randomness.

Yet, for such an approach to have important implications for economic theory, it must yield the fundamental ingredients for constrained optimization (this is needed in engineering too!) and for consistency of the resulting economic system, where consistency of the model means that it has a suitable equilibrium. Suitability means that one can define well-behaved price systems under reasonable market conditions such that at least one of these price systems can clear all markets simultaneously, given that all individual agents optimize taking prices as given. Furthermore, one wants the resulting allocations corresponding to any equilibrium to be efficient. In other words, the goal is existence and Pareto optimality of equilibrium allocations in the model. If there were possibly no equilibria or if an equilibrium could fail to be efficient in situations which otherwise satisfy appropriate versions of the well-known conditions that usually suffice to guarantee these properties, then one would naturally question the reasonableness of the proposed model.

The remainder of this paper is organized as follows: Section 2 explains several areas of engineering considerations that motivate this paper. With this motivation, Section 3 presents the proposed set of differentiated products and proves that it has the mathematical structure of a compact metric space. Section 4 presents the economic environment in terms of the new commodity space, preferences, and endowments. Then Section 5 defines competitive equilibrium, establishes its existence, and demonstrates its efficiency by appealing to a core equivalence result. Section 6 examines an alternative possible definition of differentiated products in the set $\mathcal{C}_0$ of nonempty closed convex subsets of the closed unit cube in a Euclidean space subject to production imprecision given by probabilities and explains why this approach is not adopted here. Continuing in this vein, Section 7 explores the potential re-definition of geometric objects as equivalence classes under the equivalence relations of translation or translation and rotation. Section 8 discusses various issues involved in the extension from pure exchange economies to those with production. Finally, Section 9 contains concluding comments.

2.2 Real world considerations

Geometric objects must be closed and bounded subsets of some finite-dimensional Euclidean space. Obviously, the main cases of interest are subsets of the plane and especially three-space, but $\mathbb{R}^n$ is specified in this paper because this level of added generality does not increase the difficulty. Fix a positive integer n and let $\mathcal{S}_0$ denote the set of nonempty compact subsets of $\mathbb{R}^n$. Elements of $\mathcal{S}_0$ will be called geometric objects. [Where confusion with the notion of object classes in computer science could occur, the literature uses the terms geometric solid (for three-dimensional subsets) or, more generally, artifacts, although the later terminology can be applied to virtually anything that is designed or manufactured.] Determination of the subsets of $\mathcal{S}_0$ which can be considered the natural domains of geometric objects is postponed to Subsection 2.3, after a topological structure on $\mathcal{S}_0$ has been introduced.

2.2.1 Approximations

Two distinct notions of approximation of a subset in $\mathbb{R}^n$ by a sequence (or net) of subsets in $\mathbb{R}^n$ are commonly found in the literature: the one based on the generalized volume or n-dimensional Lebesgue measure of the symmetric difference of two sets and that based on the Hausdorff metric (or, more generally, closed convergence of sets). The second choice is more natural for engineering applications and, in fact, has appeared in the engineering design literature, as discussed below.

To see the difference between the two approximation concepts, consider the problem of approximating a x cm by y cm rectangle in the plane, where $0 \leq x \leq 100$ and $0 \leq y \leq 100$, by a rectangle with integer-valued length and width (i.e., by a $\hat{x}$ cm by $\hat{y}$ cm rectangle, where $\hat{x} \in \{0, 1, \dots, 100\}$ and $\hat{y} \in \{0, 1, \dots, 100\}$). Let A be our desired set or nominal object (the x cm by y cm rectangle) and let B denote the set (the $\hat{x}$ cm by $\hat{y}$ cm rectangle) that we actually obtain as described above. Note that A and B are compact. Then the error measure based on the Hausdorff metric can be written as $\delta(A, B) = \max\big\{\max_{b \in B} \min_{a \in A} \big|\big||a - b|\big|\big|, \max_{a \in A} \min_{b \in B} \big|\big||a - b|\big|\big|\big\}$ where, for $x = (x_1, x_2) \in \mathbb{R}^2$, $\big|\big||x|\big|\big| = \max\{|x_1|, |x_2|\}$ instead of the familiar Euclidean norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ (which gives an equivalent but not identical distance between the sets A and B). The alternative area-based error measure, $\text{Area}(A \triangle B) = \text{Area}\big((A \cup B) \backslash (A \cap B)\big)$ instructs one to find the volume (or area in the plane) of the symmetric difference between the sets. It's easy to check that $\delta(A, B) = \max\{|x - \hat{x}|, |y - \hat{y}|\}$ and $\text{Area}(A \triangle B) = |x - \hat{x}| \max\{y, \hat{y}\} + |y - \hat{y}| \max\{x, \hat{x}\} - |x - \hat{x}| \cdot |y - \hat{y}|$. In this example, the $\delta(A, B)$ error measure tends to be independent of the approximate magnitudes of x and y; taking $x = y = 99.5$ and $x = y = 0.5$ both give minimum errors of 0.5. On the contrary, the area of the symmetric distance necessarily goes to zero as x and y become close to zero even though the relative errors (under either error measure) explode.

Yet another useful way to understand the differences between these two error measures is to constrast them for the following sequence of sets: the ideal desired set A is fixed and equals the square with vertices $(0, 0)$, $(0, 50)$, $(50, 50)$ and $(50, 0)$ while for each k, the set that we actually obtain is B_k, where B_k is the union of A and the rectangle with vertices $(0, 50)$, $(0, 100)$, $(1/k, 100)$, $(1/k, 50)$ so that each B_k equals A plus a vertical spike of width $1/k$. For all k, $\delta(A, B_k) = 50$ but $\text{Area}(A \triangle B_k) = 50/k \to 0$ as $k \to \infty$. This example indicates that the error measure δ is likely to perform better for certain engineering design problems than the error measure given by the volume of the symmetric difference.

2.2.2 The Hausdorff metric

The Hausdorff distance is defined for every pair of nonempty subsets of $\mathbb{R}^n$.

First, define (open) ϵ-neighborhoods of nonempty subsets of $\mathbb{R}^n$ by $B_\epsilon(A) = \{x \in \mathbb{R}^n \mid \text{there exists } y \in A \text{ with } \|x - y\| < \epsilon\}$ where $A \neq \emptyset$, $A \subseteq \mathbb{R}^n$, and $\epsilon > 0$. For every two nonempty subsets E and F of $\mathbb{R}^n$, define the (extended) Hausdorff distance $\delta(E, F)$ by $\delta(E, F) = \inf\{\epsilon \in (0, \infty] \mid E \subseteq B_\epsilon(F) \text{ and } F \subseteq B_\epsilon(E)\}$. [Say that an extended distance function, extended semimetric, or extended metric is a distance function, semimetric or metric that may assume the value of ∞.] Let $\mathcal{F}$ denote the set of subsets of $\mathbb{R}^n$ and let $\mathcal{F}_0$ denote the set of nonempty subsets of $\mathbb{R}^n$. Then the function $\delta\colon \mathcal{F}_0 \times \mathcal{F}_0 \to [0, \infty]$ is an extended semimetric on $\mathcal{F}_0$; $\delta(E, F) = 0$ whenever $E = F$, $\delta(E, F) = \delta(F, E)$, and $\delta(E, F) \leq \delta(E, G) + \delta(G, F)$. However, δ fails to be an extended metric on $\mathcal{F}_0$ because $\delta(E, F) = 0$ does not imply $E = F$; indeed $\delta(E, F) = 0$ whenever $\mathrm{cl}(E) = \mathrm{cl}(F)$, where $\mathrm{cl}(A)$ denotes the closure of A. This can be "fixed" by considering equivalence classes of sets in $\mathcal{F}_0$, where two sets are equivalent if they have the same closure. A natural representative of each equivalence class is the (unique) closed subset which equals the closure of every set contained in the given equivalence class. Of course, it simplifies the discussion to work directly with the set of nonempty closed subsets of $\mathbb{R}^n$.

The Hausdorff metric was defined by Hausdorff (1962). A convenient reference is Hildenbrand (1974, pp. 15–21), while Nadler (1978) discusses convergence of sets in greater generality. Note that the Hausdorff metric topology is closely related to the concept of closed convergence of sets; see, for instance Hildenbrand (1974). The topology induced by the Hausdorff metric has been used extensively in economic theory.

Let $\mathcal{G}$ denote the set of closed subsets of $\mathbb{R}^n$ and let $\mathcal{G}_0$ denote the set of nonempty closed subsets of $\mathbb{R}^n$. Then $\delta\colon \mathcal{G}_0 \times \mathcal{G}_0 \to [0, \infty]$ is an extended metric (since $\delta(E, F) = 0$ if and only if $E = F$ whenever $E \in \mathcal{G}_0$ and $F \in \mathcal{G}_0$) and $(\mathcal{G}_0, \delta)$ is an extended metric space. Note that the topology on $\mathcal{G}_0$ induced by the (extended) Hausdorff metric is not determined by the topology of $\mathbb{R}^n$ but rather can depend on the metric used on $\mathbb{R}^n$ in the sense that two metrics d' and d'' can define the same topology on $\mathbb{R}^n$ but induce different topologies on $\mathcal{G}_0$ unless d' and d'' are uniformly equivalent (i.e., if they yield exactly the same class of uniformly continuous real-valued functions on $\mathbb{R}^n$). This is why the above discussion specified the metric derived from the Euclidean norm on $\mathbb{R}^n$.

As mentioned above, in the context of geometric design one is concerned with closed and bounded sets. In $\mathbb{R}^n$, the closed and bounded sets are the compact sets. To set notation, let $\mathcal{S}$ be the set of compact subsets of $\mathbb{R}^n$ and let $\mathcal{S}_0$ be the set of nonempty compact subsets of $\mathbb{R}^n$. Note that $(\mathcal{S}_0, \delta)$ is a metric space; δ is a metric rather than an extended metric on $\mathcal{S}_0$ because the Hausdorff distance between any two nonempty compact sets is finite.

By a result of Aubin (1977, p. 164, Theorem 1), δ is a complete extended metric on $\mathcal{G}_0$. This says that if $\{S_k\}$ is a Cauchy sequence of sets in $\mathcal{G}_0$, then there exists $S \in \mathcal{G}_0$ such that $\lim_{k\to\infty} S_k = S$. If, in fact, $S_k \in \mathcal{S}_0$ for all k and $\delta(S_k, S) \to 0$, then S must be compact also because $\delta(T', T'') = \infty$ whenever T' is compact and

T'' is unbounded (closed but noncompact). This proves that $(\mathcal{S}_0, \delta)$ is a complete metric space.

In geometric design, one frequently works with closed sets that are contained in a given compact set because such uniform boundedness captures the notion that a maximum size initial material is available or that a given machine or manufacturing process is constrained by an overall feasible size limitation. Without loss of generality, let K denote the closed unit cube in $\mathbb{R}^n$ ($K = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1$ for all $i = 1, 2, \ldots, n\}$). Let $\mathcal{K}$ denote the set of closed subsets of K and let $\mathcal{K}_0$ denote the set of nonempty closed subsets of K so that $\mathcal{K}_0 = \{S \subseteq \mathbb{R}^n \mid S \neq \emptyset,\ S$ is closed, and $S \subseteq K\}$. Then $(\mathcal{K}_0, \delta)$ is a compact metric space. (See Hildenbrand (1974 Theorem 1, p. 17).) This property constitutes a major advantage of using the topology induced by the Hausdorff metric.

In geometric design theory, the Hausdorff topology is also applied to the boundaries of geometric solids. Implicitly, this yields another extended metric space $(\mathcal{G}_0 \backslash \{\mathbb{R}^n\}, \delta^\partial)$ and metric spaces $(\mathcal{S}_0, \delta^\partial)$ and $(\mathcal{K}_0, \delta^\partial)$. For $G, H \in \mathcal{G}_0 \backslash \{\mathbb{R}^n\}$, $G, H \in \mathcal{S}_0$, or $\mathcal{G}, H \in \mathcal{K}_0$, $\delta^\partial(G, H) = \delta(\partial G, \partial H)$, where $\partial S = \mathrm{cl}(S) \backslash \mathrm{int}(S)$ denotes the boundary of the set S. By definition, the boundary of any set in $\mathcal{S}_0$ belongs to $\mathcal{S}_0$ and the boundary of any set in $\mathcal{K}_0$ is a nonempty closed subset of the compact set K and hence belongs to $\mathcal{K}_0$. [To see that the boundary of any set in $\mathcal{S}_0$ or $\mathcal{K}_0$ must be nonempty, recall that a set is both open and closed if and only if its boundary is empty; the only subsets of the connected space $\mathbb{R}^n$ which are both open and closed are the empty set and $\mathbb{R}^n$ itself.] Note that there are sets in $\mathcal{K}_0$ that are not the boundary of any set in $\mathbb{R}^n$ (for instance, K itself). Note also that δ^∂ is not defined on all of $\mathcal{G}_0$ because $\partial \mathbb{R}^n = \emptyset$.

Observe that $(\mathcal{K}_0, \delta)$ and $(\mathcal{K}_0, \delta^\partial)$ are distinct topological spaces, although both are metric spaces. Convergence in the δ metric is not equivalent to convergence in the δ^∂ metric. To see this, for $k = 3, 4, 5, \ldots$, let $S_k = K \backslash B_{1/k}(1/2, \ldots, 1/2)$, where $B_\epsilon(x) = \big\{y \in \mathbb{R}^n \bigm| \|x - y\| < \epsilon\big\}$ denotes the open ϵ-ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ (for $\epsilon > 0$). Then $S_k \xrightarrow{\delta} K$ but $S_k \xrightarrow{\delta^\partial} K \backslash \{(1/2, \ldots, 1/2)\}$ where $K \backslash \{(1/2, \ldots, 1/2)\}$ fails to be a closed subset of K but its boundary $\partial K \cup \{(1/2, \ldots, 1/2)\}$ is closed. This example illustrates that the δ^∂-limit of a sequence of compact sets need not be a closed set. Hence $(\mathcal{K}_0, \delta^\partial)$ is not closed.

2.2.3 The domain of geometric objects

The previous subsection stated that $(\mathcal{K}_0, \delta)$ is a compact metric space when endowed with the topology induced by the Hausdorff metric. Recall that $\mathcal{K}_0$ denotes the set of nonempty closed (and automatically bounded, and therefore compact) subsets of the closed unit cube K in $\mathbb{R}^n$. Yet not all sets in $\mathcal{K}_0$ serve as appropriate geometric objects. Hence, the domain $\mathcal{D}$ of geometric objects must be a proper subset of $\mathcal{K}_0$. Of course, compactness of $\mathcal{D}$ is highly desirable for mathematical tractability.

A natural restriction on geometric objects is the requirement that they be connected sets. Indeed, if a potential geometric object is not connected, it should be considered as two or more separate geometric objects, where each one of the redefined individual geometric objects consists of a single connected component of the originally proposed geometric object. This insistence on connectedness reflects manufacturing processes and practices, in that each connected component could equally well be produced at a different facility. From an economics viewpoint, the connected components could be viewed as extreme complements in consumption if the specifics of the situation render this true for some or all consumers and, in addition, firms could consider selling the various connected components as a bundled commodity. For a familiar example, think of left gloves and right gloves.

In this paper, I proceed beyond connectedness to the stronger condition of convexity. Let $\mathcal{C}_0$ denote the set of nonempty closed (and hence compact) convex subsets of K, so that $\mathcal{C}_0 = \{S \in \mathcal{K}_0 \mid S \text{ is convex}\}$. Then $(\mathcal{C}_0, \delta)$ is a compact metric space and, in fact, $\mathcal{C}_0$ is itself a convex set under the operations of taking the (Minkowski) sum of sets and scalar multiplication; i.e., if $S, T \in \mathcal{S}_0$ and $\lambda \in \mathbb{R}$, define $S + T = \{s + t \mid s \in S \text{ and } t \in T\}$ and $\lambda S = \{\lambda s \mid s \in S\}$. See Allen (1999c) for an explicit proof.

Here I follow the research strategy of focusing on $\mathcal{C}_0$ as the domain of geometric objects because of not only the desirability of convex sets but also some problems with the interpretation of the Hausdorff metric topology when it is applied to nonconvex sets. To see an explicit example of the difficulty, define a sequence $\{S_k\}$ of nonempty compact subsets of K, where $S_1 = S_2 = K$ and for each $k = 3, 4, 5, \ldots$, $S_k = K \backslash B_{1/k}(1/2, \ldots, 1/2)$, where $B_\epsilon(x)$ denotes the open ball of radius $\epsilon > 0$ centered at x in $\mathbb{R}^n$. Then, as $k \to \infty$, $S_k \to K$ but each S_k fails to be contractable and "would not hold water" because it has a hole. Another example is provided by setting each S_k equal to the (finite) subsets of K defined by points with coordinates expressed as decimals with (at most) k digits, so that in $\mathbb{R}^2$, $S_0 = \{(0,0), (0,1), (1,0), (1,1)\}$, $S_1 = \{(s_1, s_2) \in K \mid s_1 = 0, 1/10, 2/10, \ldots 1$, and $s_2 = 0, 1/10, \ldots, 1\}$. Then, as $k \to \infty$, $S_k \to K$ even though $S_k \cap K$ does not contain an open set for any k. Clearly K and the S_k could not be viewed as close substitutes for most purposes.

One solution to this problem may be to modify or strengthen the Hausdorff topology so that it distinguishes between a set and the same set after a tiny piece has been removed. Berliant has proposed a modified Hausdorff metric for this purpose; see Berliant and Dunz (1995) and Berliant and ten Raa (1988, 1992) but note that these references alter the metric further to reflect a given set of utility functions. Current research is addressing these issues.

2.2.4 Dimensioning and tolerancing

To think about dimensioning and tolerancing (D & T), consider the goal of drilling a hole in a cube of homogeneous metal. [The hole is an example of a feature (see Shah and Mäntylä, 1995).] Three distinct criteria are involved:

(1) *Size tolerance*, which means that the radius of the hole – and its depth if it does not extend completely through the piece of metal – must be within an acceptable range, which would usually take the form of a requirement that the hole's circumference must stay entirely within an annulus defined by two concentric

circles having radii equal to the minimum value and the maximum value in the acceptable range,

(2) *Form tolerance*, which means that the hole is sufficiently circular, rather than polyhedral or oval-shaped [regardless of its size], which again is typically verified by checking that the circumference lies within an annular region, and

(3) *Position tolerance*, which requires the hole to be in approximately the correct location relative to the edges of the cube of metal or relative to the locations of other features.

The three tolerancing constraints would be tested independently and the metal would be reworked or discarded if any criterion is not satisfied. This defines a *tolerance zone* or set of acceptable geometric objects. In the literature, axioms for tolerance zones have been provided. One important aspect is that exact form cannot be required; each criterion must have some "wiggle room", which need not be symmetric.

Note that this discussion focuses on D & T standards for a single geometric object and not statistical tolerancing, in which deviations with respect to some criteria can be offset by enhanced precision in terms of other criteria. Also, statistical quality control, in which random items from a batch are inspected and then a decision is made to accept or reject the entire batch, is not considered here.

The Hausdorff metric topology has been advocated in the engineering literature (i.e., Boyer and Stewart, 1991, 1992; Requicha, 1993; Requicha and Rossignac, 1992; Stewart, 1993) as a first step toward capturing D & T standards in a mathematical model. In brief, a tolerance zone is basically defined as a (relatively) open subset of geometric objects or an open ball, in the Hausdorff metric topology, around the nominal (desired) geometric object (see also Srinivasin, 1998).

2.2.5 Some remarks on the literature

The approach taken in this paper starts from the framework of general design theory, as developed by Yoshikawa (1981), who studies topologies and filters on abstract spaces associated with engineering design. Boyer and Stewart (1991, 1992) and Stewart (1993) introduce a topology (specified by the δ^{∂} metric defined in an earlier subsection) that is related to the one studied here. Requicha (1993) and Requicha and Rossignac (1992) discuss the Boyer and Stewart metric; see also the related papers by Requicha (1980, 1983), Tilove (1980) and Tilove and Requicha (1980) that focus on regular subsets in the context of dimensioning and tolerancing. (Recall that by definition, a set is regular if it equals the closure of its interior.) My paper does not focus on regular sets; this research strategy was chosen because of the difficulties associated with using the Hausdorff metric on the space of regular sets – lack of closure and the corresponding loss of compact subsets of geometric objects – that are pointed out in Allen (1999b). Peters, Rosen, and Shapiro (1994) and Rosen and Peters (1992, 1996) propose a quite different feature-based metric space topology for spaces of regular geometric designs. [See Shah and Mäntylä (1995) for an overview of features in engineering design.] A recent article by Allen (1999a) uses the Hausdorff topology and argues that, to characterize the sets of geometric objects that are manufacturable by some process or processes, one must take limits (and this involves a convergence concept or a topology). Mathematical properties of various subspaces of geometric objects are examined in Allen (1999c), based also on the topology induced by the Hausdorff metric.

2.3 Differentiated products

Section 2 argued that, as a first approach, one could take $\mathcal{C}_0$ to be the domain of geometric objects. For reasons of intuition, consistency with dimensioning and tolerancing standards, and technical tractability, $\mathcal{C}_0$ is endowed with its topology induced by the Hausdorff metric so that $(\mathcal{C}_0, \delta)$ becomes a convex compact metric space.

Thus, subsets of $\mathcal{C}_0$ become the basic differentiated products. Notice that the statement reads "subsets *of* $\mathcal{C}_0$" rather than "subsets *in* $\mathcal{C}_0$" because a commodity is some geometric object that belongs to a specified set of geometric objects.

Let $\mathcal{D}_0$ denote the set of nonempty closed subsets of $\mathcal{C}_0$, and give $\mathcal{D}_0$ the Hausdorff metric topology derived from the Hausdorff metric topology on $\mathcal{C}_0$. Note that $\mathcal{D}_0$ is not a subset of $\mathcal{C}_0$ but rather is a collection of subsets of $\mathcal{C}_0$ so that $\mathcal{D}_0$ is a set of sets. Note also that the Hausdorff distance is invoked twice in the definition of $\mathcal{D}_0$, first in the definition of $\mathcal{C}_0$ and then in a second layer involving the convergence of nonempty closed sets of nonempty convex compact subsets of $\mathbb{R}^n$. Write $(\mathcal{D}_0, \delta)$ where no confusion can occur.

Proposition 3.1 *$\mathcal{D}_0$ is a compact metric space.*

Proof.. This follows from Theorem 1 in Hildenbrand (1974, p. 17), since $(\mathcal{C}_0, \delta)$ is a compact metric space. □

However, the discussion in Section 2.4 suggests that not all elements of $\mathcal{D}_0$ are appropriate differentiated products. For example, a set consisting of a single geometric object (a set containing just one closed convex subset of K) is obviously nonempty and closed, but it violates the principle that exact form cannot be required in dimensioning and tolerancing.

To solve this problem, $\mathcal{D}_0$ will be restricted further and a proper subset of $\mathcal{D}_0$ will be taken to be the space of differentiated products. A consequence of its definition is compactness, so that tractability is not lost. Fix $\epsilon > 0$ and let $\mathcal{D}_\epsilon$ be the subset of $\mathcal{D}_0$ such that every element of $\mathcal{D}_\epsilon$ contains an open ϵ-ball.

Proposition 3.2 *For any sufficiently small $\epsilon > 0$, $\mathcal{D}_\epsilon$ is a nonempty proper compact subset of $(\mathcal{D}_0, \delta)$.*

Proof.. If $S_k \to S$ in $(\mathcal{D}_0, \delta)$ and each S_k is a compact set containing an open ϵ-ball for the given fixed $\epsilon > 0$, then so also does S contain an open ϵ-ball. The set $\mathcal{D}_\epsilon$ is a proper subset of $\mathcal{D}_0$ whenever $\epsilon > 0$ because, for instance, singletons belong to $\mathcal{D}_0$ but not to $\mathcal{D}_\epsilon$. The set $\mathcal{D}_\epsilon$ is nonempty whenever ϵ is sufficiently small relative to the size of K. □

Notice that $\mathcal{D}_\epsilon$ is not simply the collection of closed ϵ-balls, but rather contains all subsets that contain ϵ-balls. The mapping $\{\epsilon\} \times \mathcal{C}_0 \to \mathcal{D}_0$ defined by $(\epsilon, S) \mapsto \bar{B}_\epsilon(S)$ maps onto some proper subset of $\mathcal{D}_0$. However, note that (for $\underline{\epsilon} > 0$ and $\bar{\epsilon}$ sufficiently small) the map $[\underline{\epsilon}, \bar{\epsilon}] \times \mathcal{C}_0 \to \mathcal{D}_0$ (defined as above by $(\epsilon, S) \mapsto \bar{B}_\epsilon(S)$) is continuous for the product topology derived from the topologies on $\mathbb{R}$ and $(\mathcal{C}_0, \delta)$ and the "two layer" Hausdorff metric topology on $(\mathcal{D}_0, \delta)$.

2.4 The economic environment

This section lays out the economic model. It features the set $\mathcal{D}_\epsilon$ (for some sufficiently small $\epsilon > 0$) of differentiated products defined in the previous section, where $\mathcal{D}_0 \supset \mathcal{D}_\epsilon$ was endowed with a topology.

2.4.1 The commodity space

One aspect of the economic model which has not yet been emphasized is the hypothesis that commodities in $\mathcal{D}_0$ or $\mathcal{D}_\epsilon$ are indivisible. Differentiated products are assumed to be available only in integer amounts. This is a natural assumption for geometric objects, as fraction amounts – as well as irrational quantities – are difficult to interpret in an economic context.

These indivisibilities imply that, in order to enable equilibria possibly to exist, the presence of at least one perfectly divisible good is needed. This phenomenon would arise even if $\mathcal{D}_\epsilon$ were a finite set – the effects are unrelated to the fact that the model features infinitely many distinct commodities. Desirability assumptions for the divisible good are imposed in Subsection 4.3 below (for a further discussion, see Mas-Colell, 1975, 1977).

Accordingly, let h denote the perfectly divisible (homogeneous) good. For simplicity, only one divisible good is postulated; the extension to ℓ divisible goods that are priced in equilibrium simultaneously with the pricing of the differentiated commodities is a technical exercise. See Allen (1986b) for a discussion of the mathematical difficulties and an explicit proof in the context of a more complicated model with differentiated information that can be traded on markets.

Then the *set of commodities* is $\mathcal{D}_\epsilon \cup \{h\}$, for some fixed sufficiently small $\epsilon > 0$. The *commodity space* is taken to be the set of ordered pairs of bounded integer-valued Borel (signed) measures a on $\mathcal{D}_\epsilon$ such that $|a(\mathcal{D}_\epsilon)| < \infty$ [i.e., finite sums and differences of Dirac measures on $\mathcal{D}_\epsilon$] and scalars $b \in \mathbb{R}$, where, for $d \in \mathcal{D}_\epsilon$, $a(d)$ denotes the number of units of good d in the commodity bundle, for each $d \in \mathcal{D}_\epsilon$, and $b \in \mathbb{R}$ denotes the quantity of the perfectly divisible good h. Write $c = (a, b) \in \mathcal{M}^\circ(\mathcal{D}_\epsilon) \times \mathbb{R}$ where $\mathcal{M}^\circ(\mathcal{D}_\epsilon)$ denotes the set of finite integer-valued Borel measures a on $\mathcal{D}_\epsilon$.

Let $\mathcal{M}^M(\mathcal{D}_\epsilon) = \{a \in \mathcal{M}^\circ(\mathcal{D}_\epsilon) \mid |a(\mathcal{D}_\epsilon)| \le M\}$ and let $\mathcal{M}_+^M(\mathcal{D}_\epsilon) = \{a \in \mathcal{M}^M(\mathcal{D}_\epsilon) \mid a(d) \ge 0 \text{ for all } d \in \mathcal{D}_\epsilon\}$. Then the *consumption set* for each trader in the economy is taken to be $\mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_+$ for some fixed $\epsilon > 0$ and some fixed positive finite $M \in \mathbb{R}_{++}$.

Endow $\mathcal{M}^{\circ}(\mathcal{D}_{\epsilon})$ with its weak* topology or the topology of weak convergence of measures on $\mathcal{D}_{\epsilon} \subset \mathcal{D}_0$. This is the topology of pointwise convergence on the set $\mathcal{C}(\mathcal{D}_{\epsilon})$ of continuous real-valued functions on $\mathcal{D}_{\epsilon}$; i.e., $a_k \to a$ if for every $f\colon \mathcal{D}_{\epsilon} \to \mathbb{R}$ which is continuous (and bounded because $\mathcal{D}_{\epsilon}$ is compact when endowed with the Hausdorff topology), $\int f(d)\, da_n(d) \to \int f(d)\, da(d)$. Then $\mathcal{M}^M(\mathcal{D}_{\epsilon})$ becomes a compact metric space because the weak* topology is compact and metrizable on bounded subsets. Let $d_{\bullet}$ denote a metric for $\mathcal{M}^M(\mathcal{D}_{\epsilon})$.

2.4.2 Initial endowments

Recall that $c = (a, b) \in \mathcal{M}_+^M(\mathcal{D}_{\epsilon}) \times \mathbb{R}_+$ is a commodity bundle. Designate individual endowments by the subscript zero and write $c_0 = (a_0, b_0) \in \mathcal{M}_+^M(\mathcal{D}_{\epsilon}) \times \mathbb{R}_{++}$ for an initial endowment.

To set notation, define the set of all finite integer-valued nonnegative Borel measures on $\mathcal{D}_{\epsilon}$ by $\mathcal{M}_+^{\circ}(\mathcal{D}_{\epsilon}) = \bigcup\{\mathcal{M}_+^M(\mathcal{D}_{\epsilon}) \mid M \text{ is a finite integer}\}$. The difference between these sets is that $\mathcal{M}_+^M(\mathcal{D}_{\epsilon})$ is uniformly bounded by M (i.e., $|a(\mathcal{D}_{\epsilon})| \le M$ for all $a \in \mathcal{M}_+^M(\mathcal{D}_{\epsilon})$), while $\mathcal{M}_+^{\circ}(\mathcal{D}_{\epsilon})$ consists of measures that are bounded but not uniformly so.

Where no confusion can result, the notation $c = (a, b)$ or $c_0 = (a_0, b_0)$ is used to designate either individual allocations and individual endowments or economy-wide allocations and economy-wide endowments, where "economy-wide" does not mean total or aggregate. When needed, explicit arguments are appended to c or c_0 so that $c(\cdot) = \big(a(\cdot), b(\cdot)\big)$ and $c_0(\cdot) = \big(a_0(\cdot), b_0(\cdot)\big)$ denote economy-wide allocations and endowments while, for instance, $c(i) = \big(a(i), b(i)\big)$ and $c_0(i) = \big(a_0(i), b_0(i)\big)$ refer to the allocations and endowments of some particular individual agent $i \in I$.

2.4.3 Preferences

In this economy, a preference relation $\preceq$ is a complete preorder on $\mathcal{M}_+^M(\mathcal{D}_{\epsilon}) \times \mathbb{R}_+$ [i.e., the graph $\mathrm{Gr}(\preceq)$ of $\preceq$ is a subset of $(\mathcal{M}_+^M(\mathcal{D}_{\epsilon}) \times \mathbb{R}_+) \times (\mathcal{M}_+^M(\mathcal{D}_{\epsilon}) \times \mathbb{R}_+)$] satisfying the following conditions:

(a) $\preceq$ is closed (continuity of preferences),
(b) If $c' = (a', b')$ and $c'' = (a'', b'')$ are such that $c'' \ge c'$ and $b'' > b'$, then $c'' \succ c'$ (monotonicity with strict desirability of the perfectly divisible commodity),
(c) If $c' = (a', b')$ and $c'' = (a'', b'')$ are such that $b' > 0$ and $b'' = 0$, then $c' \succ c''$ (any allocation with none of the perfectly divisible good is strictly dominated by any allocation with a positive amount of the perfectly divisible good),
(d) For any $c' = (a', b')$, there is $c'' = (a'', b'')$ with $a'' = 0$ such that $c'' \succ c'$ (yet another desirability condition for the perfectly divisible good),
(e) There is $\zeta \in \mathbb{R}$ such that if $c' = (a', b')$ and $c'' = (a'', b'')$ are such that $b' = b''$ and $d(a', a'') < 1/\zeta$ (where d denotes a metric for the weak* topology on $\mathcal{M}_+^M(\mathcal{D}_{\epsilon})$), then $(a', b' + \zeta) \succ (a'', b'')$.

Conditions (d) and (e) may be replaced by the condition (f), which is easier to understand.

(f) There exists $\zeta > 0$ such that $(0, b+\zeta) \succ (a,b)$ for all $c = (a,b) \in \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_+$.

Endow the space $\mathcal{P}$ of complete continuous preference preorders with the topology of closed convergence and let $d_{\preceq}$ be a metric for it (see Hildenbrand, 1974, for details).

The interpretation of continuity of preferences may be troublesome here, given the earlier arguments about "acceptable" sets of geometric objects and D&T notions. However, upper semicontinuity is all that is really needed, which allows for situations in which slight perturbation of a set of geometric objects results in a much worse set of geometric objects. (For example, imagine that the perturbed set contains geometric objects which must undergo costly reworking before they can be installed in an assembly line operation.)

Observe that convexity of preferences could be defined because convexity makes sense in the space $\mathcal{D}$, although convex combinations of sets in $\mathcal{D}_\epsilon$ are not the same as convex combinations of measures in $\mathcal{M}_+^\circ(\mathcal{D}_\epsilon)$ or $\mathcal{M}_+^M(\mathcal{D}_\epsilon)$. In any event, convexity is not required for the results in this paper, since a continuum of agents is needed to deal with the nonconvexities that inherently arise from the presence of indivisibilities.

2.4.4 The economy

This paper deals exclusively with large economies – those having an atomless continuum of agents. An economy then is defined to be a probability (joint) distribution on the space of preferences and endowments.

Definition 4.1 An *economy* is a Borel probability measure ν on $\big(\mathcal{P} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_{++}, \mathcal{B}(\mathcal{P} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_{++})\big)$, for some $\epsilon > 0$ sufficiently small, such that the following conditions hold: ν has compact support, $\operatorname{supp}\big(\int a_0(\cdot)\, d\nu(\cdot)\big) = \mathcal{D}_\epsilon$, and condition (e) in the definition of preferences holds uniformly for $\preceq$ in the support of the marginal distribution of ν on $\mathcal{P}$ [i.e., there is $\zeta > 0$ such that for all $\preceq$, if $c' = (a', b')$ and $c'' = (a'', b'')$ are such that $b' = b''$ and $d_\bullet(a', a'') < 1/\zeta$, then $(a', b'' + \zeta) \succ (a'', b'')$.

Remark 4.2 If, in the definition of $\mathcal{P}$, conditions (d) and (e) are replaced by condition (f), then the last requirement in Definition 4.1 can be replaced as follows: there is $\zeta > 0$ such that for every $\preceq$ in the support and every $c = (a,b) \in \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_{++}$, $(a, b+\zeta) \succ (a,b)$. This is just a uniform version of condition (f).

Remark 4.3 For the existence of competitive equilibrium result in Section 5, the last condition in Definition 4.1 can be dropped whenever each trader is hypothesized to own at most one total unit of all indivisible commodities (in $\mathcal{D}_\epsilon$) in his or her initial endowment.

Remark 4.4 Observe that the initial endowments of the perfectly divisible good are assumed to lie in some compact interval $[\underline{b}_0, \bar{b}_0]$ in $\mathbb{R}_{++}$ for almost all consumers, where $\underline{b}_0 > 0$ and $\bar{b}_0 < \infty$.

Remark 4.5 The condition that $\operatorname{supp}\big(\int a_0(\cdot)\, d\nu(\cdot)\big) = \mathcal{D}_\epsilon$ in Definition 4.1 says that all differentiated products in $\mathcal{D}_\epsilon$ (for the given ϵ) are actually available in the economy. All results remain valid if $\mathcal{D}_\epsilon$ is replaced by some smaller compact subset of $\mathcal{D}_\epsilon$. In this case, all allocations involve only differentiated products on the smaller set and only goods in the smaller set can be priced in equilibrium.

2.5 Equilibrium

As usual, an equilibrium is defined to be a price system and a feasible allocation such that each consumer's allocation is maximal (with respect to his or her preferences) on the budget set defined by the initial endowment and the price system. In this model, price systems must first be defined because the presence of infinitely many commodities usually means that, in principle, more than one candidate is available for the price space.

Accordingly, let $P = \{(p, p_b) \in \mathcal{C}(\mathcal{D}_\epsilon) \times \mathbb{R} \mid p(\cdot) \geq 0 \text{ and } p_b > 0\} = C^+(\mathcal{D}_\epsilon) \times \mathbb{R}_{++}$ define the set of price systems. This means that the price of each good is nonnegative, the price of the perfectly divisible good is strictly positive, and prices depend continuously on differentiated commodities in $(\mathcal{D}_\epsilon, \delta)$. Some zero prices for differentiated products could well arise in equilibrium because large sets in $\mathcal{D}_\epsilon$ may not be very attractive to consumers.

Definition 5.1 A Borel probability measure τ on $\big(\mathcal{P} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_{++} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_+, \mathcal{B}(\mathcal{P} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_{++} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_+)\big)$ is an *equilibrium distribution* for the economy ν if there is $p^* \in P = C^+(\mathcal{D}_\epsilon) \times \mathbb{R}_{++}$ such that:

(i) $\tau_{1,2,3} = \nu$, where $\tau_{1,2,3}$ denotes the (joint) marginal distribution of τ restricted to its first three components (the set $\mathcal{P} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_+$),
(ii) $\int c_0(\cdot)\, d\nu_{2,3}(\cdot) = \int c^*(\cdot)\, d\tau_{4,5}(\cdot)$, and
(iii) $\tau\big(\{(\succeq, c_0, c^*) \in \mathcal{P} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_{++} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_+ \mid p^*c^* \leq p^*c_0$ and if c' is such that $p^*c' \leq p^*c_0$, then $c' \preceq c^*\}\big) = 1$.

Condition (i) says that τ is a distribution corresponding to the given economy ν and condition (ii) is aggregate feasibility of the equilibrium allocation $c^*(\cdot)$. Condition (iii) requires that almost all agents maximize their preferences over their budget sets defined by p^*.

Theorem 5.2 *Any economy satisfying the assumptions in this paper has an equilibrium distribution.*

Proof.. All of the assumptions in Mas-Colell (1975) are satisfied. The proof technique involves first approximating $\mathcal{D}_\epsilon$ by an increasing sequence of finite sets (in the topology of $\mathcal{D}_\epsilon$) and obtaining a corresponding sequence of equilibrium prices and equilibrium allocations for the finite restrictions. This involves checking that individual demands are upper hemicontinuous correspondences and that the aggregate demand, for the finite restrictions, is a convex-valued upper hemicontinuous correspondence, so that Kakutani's fixed point theorem applies. Along the sequence of finite approximations, a subsequence of equilibrium distributions converges weakly (by compactness) to a distribution which one can verify is an equilibrium distribution for the original economy ν with respect to the subsequential limit of the restricted equilibrium price systems. See Allen (1986a, b) for additional details. □

Remark 5.3 The technique of examining finite approximations and taking suitable (subsequential) limits is originally due to Bewley (1972) and permeates the literature on existence of competitive equilibrium with infinitely many commodities.

Remark 5.4 Equilibrium price systems necessarily satisfy certain no arbitrage conditions. In equilibrium, the price of a set in $\mathcal{D}_\epsilon$ can never exceed the sum of disjoint sets with union equal to the original set. However, similar inequalities do not apply to set-theoretic containment.

The next goal is to obtain the First Welfare Theorem in this model. To avoid the introduction of much additional technical notation, definition of the standard concept of an efficient (or Pareto optimal) distribution for a large pure exchange economy is *not included in this version of the paper. Similarly the core is not defined formally* because its introduction requires a standard representation for the economy (see Mas-Colell, 1975).

Proposition 5.5 *A distribution τ belongs to the core of an atomless economy ν if and only if τ is an equilibrium distribution for ν.*

Proof.. See Mas-Colell (1975, Theorem 2). □

Corollary 5.6 *Any equilibrium distribution τ for an economy ν is such that the equilibrium allocation distribution $\tau_{4,5}$ is Pareto optimal for ν.*

Proof.. Core allocations are necessarily Pareto optimal. □

Remark 5.7 A direct proof of Corollary 5.6 should be possible, but one must carefully check that local nonsatiation is not violated in this model. This would avoid the necessity of introducing the mathematical concept of a standard representation.

2.6 An alternate model with probabilities

Despite the arguments in Section 2 that sets of sets are the appropriate commodities with product differentiation, one may wonder about the potential formulation and consequences of a model in which the commodities are defined to be probability distributions over some space of product characteristics or precise physical descriptions. In the context of this paper, one would replace the sets in $\mathcal{D}_\epsilon$ by probability distributions on $\mathcal{C}_0$. Note that both sets and probabilities constitute natural generalizations of singletons, which can equally well be specified by Dirac probability measures.

Let $\mathcal{Q}(\mathcal{C}_0)$ denote the space of Borel probability measures on the compact metric space $\mathcal{C}_0$. Give $\mathcal{Q}(\mathcal{C}_0)$ the weak* topology of weak convergence of probability measures. Then it becomes a compact metric space; see Parthasarathy (1967). Thus $\mathcal{Q}(\mathcal{C}_0)$ has the same mathematical properties as $\mathcal{D}_0$.

However, one problem is that deletion of the Dirac measures or, more generally the measures with atoms, results in a subset which is not closed. This implies that the D & T axiom precluding exact form cannot be accommodated easily in a probabilistic framework.

Putting aside this problem, one can proceed to consider $\mathcal{M}^\circ\left(\mathcal{Q}(\mathcal{C}_0)\right)$ as the space of probabilistically-specified differentiated commodity bundles, where again I use convex subsets of K as the domain of geometric objects for specificity. The economic interpretation is that traders buy and sell known lotteries on geometric objects.

Verification that some random realization of a geometric object was drawn from the specified probability distribution is problematic. Appealing to reputation or random testing of drawings from a given distribution and a given seller would seem to be necessary in order to justify the implicit supposition that traders know the distribution or at least have subjective distributions that are consistent and cannot be contradicted.

This approach would have the advantage of avoiding defining preferences in a derived space such as $\mathcal{D}_\epsilon$ rather than on the space $\mathcal{C}_0$ of underlying geometric objects. Continuity properties of preferences thus become more natural and intuitive. However, with uncertainty, preference relations should be replaced by cardinal utilities, as is done in Allen (1986a,b). Continuity of derived ordinal preferences for probability distributions when traders maximize expected utility should follow when the distributions are suitably dispersed, which requires more than that they be atomless.

Note that, as earlier in this paper, probability distributions are not needed for convexification since the model features an atomless continuum of agents. Moreover, the differentiated commodities defined by probabilities would still be assumed to be indivisible.

2.7 Equivalence classes of geometric objects

One might wish to refine the definition of geometric objects as differentiated products so that it reflects affine invariance. Unlike buildings and bridges, the location of a geometric object – as opposed to its delivery location – is inessential and, similarly, the orientation of a geometric object when it is delivered generally doesn't matter. This suggests that, at least for the case of geometric objects, a basic differentiated commodity should be an equivalence class (under translation and rotation in $\mathbb{R}^n$) of nonempty compact subsets of $\mathbb{R}^n$. This idea is the basis of continuing research.

2.8 Production issues

The examples of geometric objects (precision-machined metal parts and dies for plastic injection molding) naturally serve as inputs to downstream production processes more than they would be expected to be purchased as final consumption goods. When these differentiated commodities are intermediate products, the resulting demand relations must be derived from the behavior of profit-maximizing or cost-minimizing firms. The requisite upper hemicontinuity should easily follow.

Deeper production issues are associated with the production of differentiated commodities in my model. The switch from a pure exchange economy to one with production requires lower semicontinuous cost functions or, more generally, well-behaved technology sets. The lower semicontinuity of cost functions when there are multiple production processes is derived in Allen (2000).

The modern issues of manufacturability, mass customization, dedicated versus flexible tools, and agility and flexibility require more attention to the specification of production technologies or cost functions, both for the short run and the long run. Effective answers to these questions also generally require the definition of a topology on the set of differentiated products, as is argued in Allen (1999a), where a simple manufacturability problem is formally posed and analyzed.

A microeconomic model with firms usually permits the possibility of strategic behavior. Hence, game theory is needed. The customary starting point is to postulate a static noncooperative game and inspect its Nash equilibria. Here the considerations of agility and flexibility might demand at least a two-stage game, with commitments on technology choice before actual production begins. Product selection decisions are also naturally placed in a game-theoretic model.

2.9 Conclusion

The main lesson of this paper is that, at least for some purposes, economic theorists should reformulate the basic microeconomic general equilibrium model to capture the notion that actual economic commodities are subject to manufacturing imprecision. In practice, this means that consumers and firms cannot guarantee that they purchase a product satisfying a complete and exact physical description, but rather they purchase an item that belongs to some specified set of products, where the set must permit some nontrivial range of all aspects of the product. The standard model can be modified and extended to take into account these considerations and nevertheless remain useful for economic theory in the sense that major results on the existence and efficiency of competitive equilibrium stay valid in the proposed reformulation.

The same principles can be applied to situations in which the underlying basic differentiated commodities are not restricted to be geometric objects. One simply requires a compact metric space of base commodities where the metric is compatible not only with consumers' notions of the substitution possibilities among goods but also with the relevant dimensioning and tolerancing standards and technological feasibilities.

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