The Impact of a Hausman Pretest on the Size of Hypothesis Tests

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Abstract

This paper investigates the size properties of a two-stage test in the linear instrumental variables model when in the first stage a Hausman (1978) specification test is used as a pretest of exogeneity of a regressor. In the second stage, a simple hypothesis about a component of the structural parameter vector is tested, using a t-statistic that is based on either the ordinary least squares (OLS) or the two-stage least squares estimator depending on the outcome of the Hausman pretest. The asymptotic size of the two-stage test is derived in a model where weak instruments are ruled out by imposing a lower bound on the strength of the instruments. The asymptotic size is a function of this lower bound and the pretest and second stage nominal sizes. The asymptotic size increases as the lower bound and the pretest size decrease and is close to or equal to 1 for empirically relevant scenarios. As a further result, it is shown that, asymptotically, the conditional size of the second stage test, conditional on the pretest not rejecting the null of regressor exogeneity, is 1 or very close to 1 even for a large lower bound on the strength of the instruments. The size distortion is caused by a discontinuity of the asymptotic distribution of the test statistic in the correlation parameter between the structural and reduced form error terms. The Hausman pretest does not have sufficient power against correlations that are local to zero while the OLS t-statistic takes on large values for such nonzero correlations.

Keywords: asymptotic size, Hausman specification test, pretest, size distortion

JEL Classification Numbers: C12

1 Introduction

This paper is concerned with the asymptotic size properties of a two-stage test where in the first stage, a Hausman (1978) specification test is used as a pretest.[1] As the lead example, the pretest tests exogeneity of a regressor in a linear instrumental variables (IV) model. In the second stage, a hypothesis about a component of the structural parameter vector is tested using a t-statistic based on either the ordinary least squares (OLS) or the two-stage least squares (2SLS) estimator, depending on the outcome of the pretest. An explicit formula for the asymptotic size of the two-stage test is derived in a model where weak instruments are ruled out by imposing a lower bound on the strength of the instruments. The asymptotic size is a function of the nominal size of the pretest, the nominal size of the second stage test, the number of instruments, and the lower bound on the strength of the instruments.

It is known that pretesting may impact the size properties of two-stage tests. For example, Kabaila (1995), Andrews and Guggenberger (2005e, AG henceforth), and Leeb and Pötscher (2005) discuss confidence intervals (CIs) based on an estimator that can be viewed as a post-model-selection estimator based on a consistent model selection procedure. They show that the CI has asymptotic confidence size equal to 0. AG (2005b) considers tests concerning a parameter in a linear regression model after a "conservative" model selection procedure has been applied to determine whether another regressor should enter the model. They find that the two-stage test is extremely size distorted. However, to the best of my knowledge, no results are available regarding the impact of the Hausman pretest on the size of a two-stage test.

A Monte Carlo study in the linear IV model assesses the finite sample size properties of the two-stage test that uses the Hausman pretest in the first stage. An array of empirically relevant parameter choices is used for the concentration parameter μ^2 and the correlation between structural and reduced form error ρ. Hansen, Hausman, and Newey (2004) provide estimates of μ^2 and ρ from data sets in recently published applied papers in several top journals. Of the data sets they consider, the first and third quartiles of the estimated concentration parameter are 13 and 105 and the first and third quartiles of the estimated correlation are .07 and .47. For sample size $n = 1000$, 5 instruments, nominal sizes of the pretest and second stage test equal to .05, the finite sample null rejection probabilities of the two-stage test equal .87, .91, .72, .74, .15, .06 when (μ^2, ρ) equals (13,.1), (13,.3), (13,.5), (113,.1), (113,.3), and (113,.5),

[1]The specification tests proposed in Hausman's (1978) seminal paper are routinely used as pretests in applied work, see e.g. Bradford (2003). As of November 2007, *www.jstor.org* lists about 450 citations of Hausman (1978). This number is likely a lower bound on the number of applied papers that use Hausman tests as pretests because many applied papers that use a Hausman test do so, without explicitly citing Hausman (1978) in the references. In the *American Economic Review* alone (until 2004) there are at least 75 applied papers that use a Hausman test (about 25 of these papers were written in the years 2000-2004). Many of these papers did not cite Hausman (1978).

Oftentimes, these specification tests are also referred to as *Durbin-Wu-Hausman tests* based on the papers by Durbin (1954), Wu (1973), and Hausman (1978).

respectively. On the other hand, a simple t-test based on the 2SLS estimator has null rejection probabilities equal to .01, .06, .15, .04, .05, and .07 for these cases and thus virtually uniformly dominates the size distorted two-stage procedure in terms of null rejection properties.

The paper then develops the theory to confirm the simulation results by deriving an explicit formula for the asymptotic size of the two-stage test under strong instruments asymptotics.[2] The asymptotic size of the two-stage test increases as the lower bound on the instrument strength, denoted by κ, or the pretest size decrease. It is equal to 1 or close to 1 for empirically relevant scenarios. For example, for a pretest and second stage test nominal size of 5% and $\kappa = .001$ or .1, the asymptotic size of the symmetric two-sided test equals 1.00 and .95, respectively. For comparison, note that for the Angrist and Krueger (1991) data the strength of the instruments equals .017 and .028 for the setup with 3 and 180 instruments, respectively. See below for further discussion of this example. The result on the asymptotic size of the two-stage test, denoted by $AsySz(\theta_0)$, immediately implies an upper bound on the asymptotic confidence size of confidence intervals, obtained by inverting the two-stage test, given by $1 - AsySz(\theta_0)$.

As another main result, it is shown that the conditional size of the two-stage test, conditional on the Hausman pretest not rejecting the null hypothesis of exogeneity, equals 1 or is close to 1 in empirically relevant scenarios.

Sequences of nuisance parameters are characterized that lead to the highest null rejection probabilities of the two-stage test asymptotically. For sequences of correlations ρ that are local to zero of order $n^{-1/2}$, the Hausman pretest statistic converges to a noncentral chi-squared distribution. The noncentrality parameter is small when the strength of the instruments is small. In this situation, the Hausman pretest has low power against local deviations of the pretest null hypothesis and consequently, with high probability, OLS based inference is done in the second stage. However, the second stage OLS based t-statistic may take on very large values under such local deviations. The latter causes size distortion in the two-stage test. If, on the other hand, ρ is kept fixed as n goes off to infinity, then the two-stage procedure has good asymptotic null rejection probabilities: If ρ is nonzero, the Hausman pretest statistic diverges to infinity, and in the second stage a 2SLS based t-statistic is used. In this case, the asymptotic null rejection probability of the two-stage test equals the nominal size. If ρ equals zero, the Hausman pretest statistic converges to a central chi-squared distribution and therefore with probability equal to $1 - \beta$ (where β denotes the nominal size of the pretest) a t-statistic based on the OLS estimator is used in the second stage. Because $\rho = 0$, the asymptotic null rejection probability

[2] Intuitively, the terminology "strong" can be interpreted as a situation where the reduced form coefficient matrix is fixed and has full rank. In the scalar situation, it essentially means that the correlation between the instrument and the included endogenous variable is bounded away from zero. The precise definition, in the notation of (2.10), is that $\gamma_2 = ||(\Omega^{1/2}\pi/\sigma_v|| = (\mu^2/n)^{1/2}$ is bounded away from zero, i.e. $\gamma_2 \geq \kappa$ for some lower bound on the instrument strength $\kappa > 0$.

of the OLS based t-test equals the nominal size. With probability β, a t-test based on the 2SLS estimator is used in the second stage whose asymptotic null rejection probability equals the nominal size. However, this heuristic pointwise justification of the two-stage procedure does not hold uniformly and the asymptotic size of the test is 1 for empirically relevant values of κ.

Note that in the "strong instrument scenario" considered here, a 2SLS based t-statistic has correct asymptotic size while the two-stage procedure is severely size distorted in empirically relevant scenarios. If inference on the structural parameter is the object of interest and the researcher is concerned about the null rejection probability of the inference procedure, the above findings suggest that it is not prudent to mechanically implement a Hausman test as a pretest. On the other hand, simply using a 2SLS based t-statistic is theoretically justified.[3] Guggenberger (2007) studies the asymptotic size properties of the two-stage test when weak instruments are allowed for, i.e. $\kappa = 0$. When weak instruments are not excluded, the space of nuisance parameters is larger, and therefore, it is not surprising that the asymptotic size of the two-stage test equals 1.

Next, the related literature is discussed. This paper is closely related to the sequence of papers AG (2005a-e). As in these papers, size distortion arises here because the test statistic has an asymptotic distribution that is discontinuous in nuisance parameters of the model. The discontinuity in the present case arises when there is zero correlation between the structural and reduced form error terms.

This paper is related to the papers by Hahn and Hausman (2002) and Hausman, Stock, and Yogo (2005). The former paper suggests a Hausman-type (pre-)test of the null hypothesis of instrument validity. The latter paper shows that a second stage Wald test is equally size distorted unconditionally and conditional on the Hahn and Hausman (2002) pretest not rejecting the null hypothesis of strong instruments. Another paper that is concerned with the size effects of pretests is Hall, Rudebusch, and Wilcox (1996). They investigate by Monte Carlo simulation the conditional and unconditional null rejection probabilities of a second stage t-test, if in the first stage the sample correlation between regressors and instruments is used as a pretest for instrument relevance. They find that the conditional size properties of the t-test, conditional on the pretest rejecting the null of instrument irrelevance, are not better than the unconditional size properties. Dhrymes (2003) and papers cited therein provide modified versions of Hausman pretests.

Next, other common applications of Hausman specification tests as pretests are discussed. The recent paper by Hausman and White (2006) provides a more detailed

[3] If, in addition, instruments are potentially weak, that is, the strength of the instruments is not bounded away from zero, my recommendation is to use any of the robust testing procedures suggested by Anderson and Rubin (1949), Moreira (2001, 2003), Kleibergen (2002), Guggenberger and Smith (2005), and Andrews, Moreira, and Stock (2006). In situations, where very weak instruments can be excluded and many instruments are used, the modified Wald test in Newey and Windmeijer (2004) can be applied.

overview. The results described above strongly suggest that similar size problems arise for all these applications. First, Hausman pretests have been suggested to test for exogeneity of potential instruments. Staiger and Stock (1997) shows size distortion of the standard Hausman pretest under weakness of instruments and Hahn, Ham, and Moon (2007) introduces a modified version of the Hausman pretest that is robust to weak instruments. They do not however investigate the size properties of the two-stage test which is the focus of this paper. In Guggenberger (2007), it is shown that the conditional size of the two-stage test, conditional on the pretest not rejecting, is 1. Second, in a panel data context, under independence of the regressors and individual specific effects, the random effects estimator is consistent and efficient but inconsistent otherwise. On the other hand the fixed effect estimator is consistent even if the independence assumption fails. Third, in a system of linear simultaneous equations, three-stage least squares is consistent and efficient for estimation of the first equation under correct specification of all equations, but typically inconsistent otherwise while 2SLS is consistent if the first equation is correctly specified.

The remainder of the paper is organized as follows. Subsections 2.1 and 2.2 describe the model and test statistic. Subsection 2.3 reports finite sample results using empirically relevant parameter choices. The remainder of Section 2 derives the asymptotic size results of the two-stage test when the Hausman pretest is used to test for exogeneity of a regressor.[4]

2 The Size of Tests After a Hausman Pretest

This section deals with the asymptotic size of the two-stage test in the linear IV model where in the first stage the Hausman pretest tests for exogeneity of a regressor.

2.1 Model and Definitions

Consider the linear IV model

$$\begin{aligned} y_1 &= y_2\theta + X\zeta + u, \\ y_2 &= Z\pi + X\phi + v, \end{aligned} \tag{2.1}$$

where $y_1, y_2 \in R^n$, $X \in R^{n \times k_1}$ for $k_1 \geq 0$ is a matrix of exogenous variables, $Z \in R^{n \times k_2}$ for $k_2 \geq 1$ is a matrix of IVs, and $(\theta, \zeta', \phi', \pi')' \in R^{1 \times k_1 \times k_1 \times k_2}$ are unknown parameters.

[4] A Supplementary Appendix, Guggenberger (2007), discusses several additional results. It shows that, for a given bound on the instrument strength, the size correction methods of Andrews and Guggenberger (2005b) could be applied to size-correct the two-stage test. It shows that, if one allows for weak instruments, the asymptotic size of the two-stage test is 1 and size-correction is not possible. It discusses subsampling versions of the test. It shows that the same size problems of two-stage tests arise in other applications of a Hausman pretest, for example, when it is used to test for instrument exogeneity. Finally, additional Monte Carlo results are given, including power results for the simulations in Section 2.3.

Let $\overline{Z} = [X:Z]$ and $k = k_1 + k_2$. For $j = 1, 2$, denote by $y_{j,i}$, u_i, v_i, X_i, Z_i, and $\overline{Z}_i$ the i-th rows of y_j, u, v, X, Z, and $\overline{Z}$, respectively, written as column vectors (or scalars). The observed data are y_1, y_2, X, and Z. The data $(u_i, v_i, \overline{Z}_i)$, $i = 1, ..., n$, are i.i.d.

The paper investigates the asymptotic size of a two-stage test of the null hypothesis

$$H_0 : \theta = \theta_0 \tag{2.2}$$

where in the first stage a Hausman (1978) test is undertaken as a pretest. One- and two-sided alternatives are considered.

The Hausman pretest tests exogeneity of the variable $y_{2,i}$.[5] If the pretest rejects the exogeneity hypothesis, then, in the second stage, $H_0 : \theta = \theta_0$ is tested by using a t-test based on the 2SLS estimator. If the pretest does not reject the exogeneity hypothesis, a t-test based on the OLS estimator is used in the second stage.

Denote by α and β the nominal sizes of the second stage and first stage test. To my knowledge, it has not been discussed in the literature what the resulting asymptotic null rejection probability of the two-stage test is as a function of α and β, even under the assumption of strong identification and fixed ρ (in particular, $\rho = 0$), let alone its asymptotic size. To derive the resulting asymptotic null rejection probability under these assumptions is not hard and only requires deriving the joint distribution of the pretest statistic and the possible second stage statistics. In this section, a formula for the asymptotic size of the two-stage test is derived. By definition, the asymptotic size of a test of the null hypothesis $H_0 : \theta = \theta_0$ in the presence of nuisance parameters $\gamma \in \Gamma$ equals

$$AsySz(\theta_0) = \limsup_{n\to\infty} ExSz_n(\theta_0), \tag{2.3}$$

where

$$ExSz_n(\theta_0) = \sup_{\gamma\in\Gamma} RP_n(\theta_0, \gamma),\ \ RP_n(\theta_0, \gamma) = P_{\theta_0,\gamma}(T_n(\theta_0) > c_{1-\alpha}), \tag{2.4}$$

$T_n(\theta_0)$ is the test statistic, $c_{1-\alpha}$ the critical value of the test, and $P_{\theta,\gamma}(\cdot)$ denotes probability when the true parameters are (θ, γ). The test statistics $T_n(\theta_0)$, critical values $c_{1-\alpha}$, and parameter space Γ for the present application are defined in the next subsections. The parameter space is modelled as a function of the strength of the instruments in subsection 2.4.

See AG (2005a) and Section 2 in AG (2005d) for a detailed discussion of uniformity and the important distinction between pointwise null rejection probability and size. Uniformity over $\gamma \in \Gamma$ which is built into the definition of $AsySz(\theta_0)$ is crucial for the asymptotic size to give a good approximation for the finite sample size.

[5] Hillier (1987) and Moreira (2001, p.7 of the July 2005 revision of the paper) provide an interesting discussion of the connection between structural parameter tests and exogeneity tests.

2.2 Test Statistics and Critical Values

In this subsection the two-stage test statistic $T_n(\theta_0)$ for the hypothesis test $H_0 : \theta = \theta_0$ is defined. Denote by I_n the n-dimensional identity matrix. For a matrix W with n rows, define $P_W = W(W'W)^{-1}W'$, $M_W = I_n - P_W$, and $W^{\perp} = M_X W$ and, if no X appears in (2.1), set $W^{\perp} = W$.

The Hausman pretest is defined as

$$H_n = \frac{n(\widehat{\theta}_{2SLS} - \widehat{\theta}_{OLS})^2}{\widehat{V}_{2SLS} - \widehat{V}_{OLS}}, \tag{2.5}$$

where

$$\begin{aligned}
\widehat{\theta}_{2SLS} &= y_2' P_{Z^{\perp}} y_1 / (y_2' P_{Z^{\perp}} y_2), \\
\widehat{\theta}_{OLS} &= y_2' M_X y_1 / (y_2' M_X y_2), \\
\widehat{V}_{2SLS} &= (y_2' P_{Z^{\perp}} y_2 / n)^{-1} \widehat{\sigma}_u^2(\widehat{\theta}_{2SLS}), \\
\widehat{V}_{OLS} &= (y_2' M_X y_2 / n)^{-1} \widehat{\sigma}_u^2(\widehat{\theta}_{OLS}), \text{ and} \\
\widehat{\sigma}_u^2(\widehat{\theta}_l) &= n^{-1}(y_1^{\perp} - y_2^{\perp}\widehat{\theta}_l)'(y_1^{\perp} - y_2^{\perp}\widehat{\theta}_l)
\end{aligned} \tag{2.6}$$

for $l = OLS$ and $2SLS$. Other definitions of H_n are possible, that replace $\widehat{\sigma}_u^2(\widehat{\theta}_{OLS})$ by $\widehat{\sigma}_u^2(\widehat{\theta}_{2SLS})$ or vice versa. The results on the asymptotic size do not depend on which definition is used, see (2.18) below. If y_2 is exogenous and the instruments are strong then $H_n \rightarrow_d \chi_1^2$ as $n \rightarrow \infty$ under assumptions given in Hausman (1978).

Define the t-test statistic

$$T_l^*(\theta) = n^{1/2}(\widehat{\theta}_l - \theta)/\widehat{V}_l^{1/2} \tag{2.7}$$

for $l = OLS$ and $2SLS$. The standard definition of the two-stage test statistic is

$$T_n^*(\theta_0) = T_{OLS}^*(\theta_0) I(H_n \leq \chi_1^2(1-\beta)) + T_{2SLS}^*(\theta_0) I(H_n > \chi_1^2(1-\beta)), \tag{2.8}$$

where, again, β is the nominal size of the pretest, I is the indicator function, and $\chi_1^2(1-\beta)$ the $1-\beta$ quantile of a chi-square random variable with one degree of freedom. Define the two-stage test statistic $T_n(\theta_0)$ as $\pm T_n^*(\theta_0)$ or $|T_n^*(\theta_0)|$ depending on whether the test is a lower/upper one-sided or a symmetric two-sided test, respectively.

The nominal $1 - \alpha$ standard fixed critical value (FCV) test rejects H_0 if

$$T_n(\theta_0) > c_{\infty}(1 - \alpha), \tag{2.9}$$

where $c_{\infty}(1 - \alpha) = z_{1-\alpha}$, $z_{1-\alpha}$, and $z_{1-\alpha/2}$ for the upper one-sided, lower one-sided, and symmetric two-sided test, respectively and $z_{1-\alpha}$ is the $1-\alpha$ quantile of a standard normal distribution.

2.3 Finite Sample Evidence

Next, the finite sample size properties of the two-stage test are investigated in a simulation study based on parameter choices for the concentration parameter $\mu^2 = n\pi' EZ_iZ_i'\pi/Ev_i^2$ and the correlation $\rho = Corr(u_i, v_i)$ that were estimated from data sets in applied papers published in the last five years in the *American Economic Review* (AER), *Journal of Political Economy* (JPE), and the *Quarterly Journal of Economics* (QJE), see Hansen, Hausman, and Newey (2004).[6] Their Table 7 is reproduced here; it reports several percentiles Q10, ..., Q90 for the concentration and correlation parameters in these data sets:

Hansen, Hausman, and Newey (2004), Table 7

Five years of AER, JPE, and QJE

	# of papers	$Q10$	$Q25$	$Q50$	$Q75$	$Q90$
μ^2	28	8.95	12.7	23.6	105	588
ρ	22	.022	.0735	.279	.466	.555

In the simulations, the nominal sizes of the pretest and the second stage test are $\alpha = \beta = .05$. Furthermore, $EZ_iZ_i' = I_{k_2}$ and $Ev_i^2 = 1$. This implies $||\pi|| = \sqrt{\mu^2}n^{-1/2}$. The vector π is chosen to have all components equal, $\pi = \pi_0(1, ..., 1)' \in R^{k_2}$ for $\pi_0 \in R$. The vector (u_i, v_i, Z_i) is chosen as i.i.d. normal with zero mean and unit variances and Z_i is independent of u_i and v_i. The asymptotic results do not depend on k_1, the number of included exogenous variables, and therefore $k_1 = 0$ in the simulations.

Two Monte Carlo experiments based on the information in Table 7 of Hansen, Hausman, and Newey (2004) are implemented.

In the first experiment, the values of μ^2 and ρ are fixed at the estimated median values over the data sets, namely $\mu^2 = 23.6$ and $\rho = .279$. Empirical null rejection probabilities of the two-stage test are reported for various values of the sample size n and the number of instruments k_2, namely $n \in \{100, 1000, 10000\}$ and $k_2 = \{1, 5, 20\}$. In Table Ia below, columns 4 and 5 with headings "Upper" and "Sym" report these finite sample null rejection probabilities for upper and symmetric two-stage tests. Column 6 with heading "HPre" reports null rejection probabilities of the Hausman pretest. Finally, columns 7 and 8 with headings "CondlUpper" and "CondlSym" report conditional probabilities of rejecting the null hypothesis of the second stage test, conditional on the Hausman pretest not rejecting the pretest null hypothesis.

For all configurations, the two-stage test overrejects severely, with null rejection probabilities in the range $[.62, .85]$. The pretest null hypothesis is only rejected with probabilities ranging roughly between 10% and 20% even though $\rho = .279$. However, conditional on not rejecting the pretest null hypothesis and thus using an OLS based

[6] The concentration parameter μ^2 equals $n\pi' EZ_iZ_i'\pi/Ev_i^2$ when there are no included exogenous variables. In general, the concentration parameter is defined as $n\gamma_2^2$ where γ_2 is defined in (2.10).

t-statistic in the second stage, the null rejection probabilities equal 100% in most scenarios. The OLS based t-statistic takes on very large values under the failure of the pretest null hypothesis while the Hausman pretest does not.

Insert Table Ia about here

In the second experiment, the sample size and the number of instruments are fixed at $n = 1000$, $k_2 = 5$ and various values of the concentration parameter μ^2 and ρ are considered that cover the whole range of values reported in Hansen, Hausman, and Newey (2004), namely $\mu^2 \in \{0, 13, 50, 113, 200, 313, 450, 613\}$ and $\rho \in \{0, .05, .1, .2, .3, .4, .5, .6\}$. Therefore, the results cover all the cases of combinations of μ^2 and ρ that were found in the applied papers in the last five years in AER, JPE, and QJE considered in the table above. For each such combination, Table Ib below reports null rejection probabilities of the symmetric two-stage test and of the symmetric t-test based on the 2SLS estimator. The results strongly suggest that in terms of null rejection probabilities, simply using the one-stage t-test, is the better of the two methods. In situations, where the two-stage test has good null rejection probabilities (the cases where $\rho = 0$ or ($\rho \geq .3$ and $\mu^2 \geq 200$)), the same is true for the one-stage t-test. However, in all other situations the two-stage test overrejects, oftentimes severely, while the one-stage test has relatively good size properties (except when $\rho \geq .5$ and $\mu^2 \leq 13$). For example, for the cases ($.1 \leq \rho \leq .4$ and $\mu^2 \leq 13$) the null rejection probabilities of the two-stage test fall into the interval $[.84, 1.00]$ while the corresponding interval for the one-stage test is $[0, .1]$. For $\rho = .1$ the null rejection probability of the two-stage test is .87 when $\mu = 13$ and .38 when $\mu = 613$ while for the one-stage test, the corresponding probabilities are .01 and .05.

Insert Table Ib about here

In the next subsections, the theoretical evidence is provided to support the results of the finite sample simulations. The next subsection defines the space of nuisance parameters. Finally, the asymptotic size of the two-stage test is derived.

2.4 Parameter Space

In this subsection, the parameter space Γ of the nuisance parameter vector γ is defined. Following AG (2005a), the parameter γ has three components: $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. The points of discontinuity of the asymptotic distribution of the test statistic of interest are determined by the first component, γ_1. The parameter space of γ_1 is Γ_1. The second component, γ_2, of γ also affects the limit distribution of the test statistic, but does not affect the distance of the parameter γ to the point of discontinuity. The parameter space of γ_2 is Γ_2. The third component, γ_3, of γ does not affect the limit distribution of the test statistic. The parameter space for γ_3 is $\Gamma_3(\gamma_1, \gamma_2)$, which generally may depend on γ_1 and γ_2.

The "strength of the instruments", $||(\Omega^{1/2}\pi/\sigma_v||$, affects the limit distribution of the test statistics discontinuously at the point 0 of no identification, see Guggenberger (2007, Section 6). Because the data evidence in Hansen, Hausman, and Newey (2004) suggests that extremely weak identification is rather the exception, a lower bound on the strength of the instruments $||(\Omega^{1/2}\pi/\sigma_v|| \geq \kappa$ is imposed for some $\kappa > 0$. Weak instruments as in Staiger and Stock (1997), that would correspond to $\kappa = 0$, are therefore ruled out. By imposing a lower bound, $||(\Omega^{1/2}\pi/\sigma_v||$ no longer affects the limit distribution discontinuously, but continuously, see below.

Assume that $\{(u_i, v_i, X_i, Z_i) : i \leq n\}$ are i.i.d. with distribution F. Define the vector of nuisance parameters $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, by

$$\begin{aligned}
&\gamma_1 = \rho,\ \gamma_2 = ||(\Omega^{1/2}\pi/\sigma_v||,\ \text{and } \gamma_3 = (F, \pi, \zeta, \phi),\ \text{where} \\
&\sigma_u^2 = E_F u_i^2,\ \sigma_v^2 = E_F v_i^2,\ \rho = Corr_F(u_i, v_i), \\
&\Omega = Q_{ZZ} - Q_{ZX}Q_{XX}^{-1}Q_{XZ},\ \text{and } Q = \begin{bmatrix} Q_{XX} & Q_{XZ} \\ Q_{ZX} & Q_{ZZ} \end{bmatrix} = E_F \overline{Z}_i \overline{Z}_i', \qquad (2.10)
\end{aligned}$$

and $||\cdot||$ denotes Euclidean norm. The parameter γ_1 measures the degree of endogeneity of y_2.[7] The parameter γ_2 measures the strength of the instruments. It is related to the concentration parameter μ^2 (defined above for the particular case $k_1 = 0$) by $\gamma_2 = n^{-1/2}\mu$. Let

$$\Gamma_1 = [-1, 1],\ \Gamma_2 = [\kappa, \overline{\kappa}] \qquad (2.11)$$

for some $0 < \kappa < \overline{\kappa} < \infty$. The technical details of the definition of $\Gamma_3 = \Gamma_3(\gamma_1, \gamma_2)$ are given in the Appendix, see (3.1). Finally, define the parameter space Γ of γ as

$$\Gamma = \{\gamma = (\gamma_1, \gamma_2, \gamma_3) : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2, \gamma_3 \in \Gamma_3(\gamma_1, \gamma_2)\}. \qquad (2.12)$$

2.5 Asymptotic Distributions and Size

In this subsection, the asymptotic distribution of the test statistic is derived under certain parameter sequences $\{\gamma_{n,h}\}$ defined below. Then the asymptotic size of the test is determined.

Let $R_\infty = R \cup \{\pm\infty\}$. Define

$$\begin{aligned}
H &= \{h = (h_1, h_2) \in R_\infty^2 : \exists\ \{\gamma_n = (\gamma_{n,1}, \gamma_{n,2}, \gamma_{n,3}) \in \Gamma : n \geq 1\} \\
&\quad \text{such that } n^{1/2}\gamma_{n,1} \to h_1 \text{ and } \gamma_{n,2} \to h_2\}. \qquad (2.13)
\end{aligned}$$

It follows that

$$H = H_1 \times H_2 = R_\infty \times [\kappa, \overline{\kappa}]. \qquad (2.14)$$

[7]Note that in AG (2005a-e) the specification for γ has always been chosen such that when γ_1 times n^r diverges to infinity, the "standard FCV" asymptotic distribution is obtained. In this example, when $n^{1/2}|\gamma_1| \to \infty$, y_2 is not exogenous. Instead, the "standard" Hausman (1978) result $H_n \to_d \chi_1^2$ is obtained under $n^{1/2}|\gamma_1| \to 0$ and additional assumptions.

Two cases are dealt with separately. Case I has $|h_1| < \infty$ while Case II has $|h_1| = \infty$. In Case I, $\rho \to 0$ and thus $var(u_i v_i)/(\sigma_u^2 \sigma_v^2) \to 1$, see (3.2). In Case I, y_2 is only "weakly endogenous" while in Case II it is "strongly endogenous".

Definition of $\{\gamma_{n,h}\}$: For $h = (h_1, h_2) \in H$, let $\{\gamma_{n,h}\} \subset \Gamma$ denote a sequence of parameters with components $\gamma_{n,h,1}, \gamma_{n,h,2}$, and $\gamma_{n,h,3}$, $\gamma_{n,h,1} = (\gamma_{n,h,1}, \gamma_{n,h,2}, \gamma_{n,h,3})'$, where

$$\begin{aligned} \gamma_{n,h,1} &= Corr_{F_n}(u_i, v_i),\ \gamma_{n,h,2} = ||(\Omega_n^{1/2}\pi_n/(E_{F_n}v_i^2)^{1/2}||, \text{ for} \\ \Omega_n &= E_{F_n}Z_iZ_i' - E_{F_n}Z_iX_i'(E_{F_n}X_iX_i')^{-1}E_{F_n}X_iZ_i', \text{ s.t.} \\ n^{1/2}\gamma_{n,h,1} &\to h_1,\ \gamma_{n,h,2} \to h_2, \text{ and } \gamma_{n,h,3} = (F_n, \pi_n, \zeta_n, \phi_n) \in \Gamma_3(\gamma_{n,h,1}, \gamma_{n,h,2}). \end{aligned} \tag{2.15}$$

As Theorem 1 below shows, the highest asymptotic null rejection probability of the test is realized along some sequence of the type $\{\gamma_{n,h}\}$. It is therefore enough to study the asymptotic rejection rates along sequences $\{\gamma_{n,h}\}$. Under any sequence $\{\gamma_{n,h}\}$ for which $Corr_{F_n}(u_i, v_i) \to \rho$, the following convergence result holds

$$\begin{aligned} &\begin{pmatrix} (n^{-1}Z^{\perp\prime}Z^{\perp})^{-1/2}n^{-1/2}Z^{\perp\prime}u/\sigma_u \\ (n^{-1}Z^{\perp\prime}Z^{\perp})^{-1/2}n^{-1/2}Z^{\perp\prime}v/\sigma_v \\ n^{-1/2}(u'v - E_{F_n}u'v)/(\sigma_u\sigma_v) \end{pmatrix} \to_d \begin{pmatrix} \psi_{u,\rho} \\ \psi_{v,\rho} \\ \psi_{uv,\rho} \end{pmatrix} \\ &\sim N(0, \begin{pmatrix} V_\rho \otimes I_{k_2} & 0 \\ 0' & 1+\rho^2 \end{pmatrix}) \text{ for } V_\rho = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \end{aligned} \tag{2.16}$$

where $\psi_{u,\rho}, \psi_{v,\rho} \in R^{k_2}$, $\psi_{uv,\rho} \in R$. See AG (2005c, eq. (2.15)) for similar statements.[8]

Next the limit distribution of the test statistic $T_n^*(\theta_0)$ is derived under sequences $\gamma_{n,h}$. To do so, (2.16) and derivations from AG (2005c, Sections 2.3 and 4.1.2) are used. For Case I and $\xi_h = (\xi_{1,h}, ..., \xi_{4,h})'$, $h = (h_1, h_2)'$

$$\begin{pmatrix} n^{-1/2}y_2'P_{Z^\perp}u/(\sigma_u\sigma_v) \\ n^{-1/2}y_2'M_Xu/(\sigma_u\sigma_v) \\ n^{-1}y_2'P_{Z^\perp}y_2/\sigma_v^2 \\ n^{-1}y_2'M_Xy_2/\sigma_v^2 \end{pmatrix} \to_d \xi_h \sim \begin{pmatrix} h_2 s_{k_2}'\psi_{u,0} \\ h_2 s_{k_2}'\psi_{u,0} + \psi_{uv,0} + h_1 \\ h_2^2 \\ h_2^2 + 1 \end{pmatrix}, \tag{2.17}$$

where $s_{k_2} \in R^{k_2}$ is an arbitrary vector with $||s_{k_2}|| = 1$. Therefore,

$$\begin{pmatrix} T_{2SLS}^*(\theta_0) \\ T_{OLS}^*(\theta_0) \\ H_n \\ \widehat{\sigma}_u^2(\widehat{\theta}_{2SLS})/\sigma_u^2 \\ \widehat{\sigma}_u^2(\widehat{\theta}_{OLS})/\sigma_u^2 \end{pmatrix} \to_d \eta_h = \begin{pmatrix} s_{k_2}'\psi_{u,0} \\ (1+h_2^2)^{-1/2}\xi_{2,h} \\ (1+h_2^2)[s_{k_2}'\psi_{u,0} - h_2(1+h_2^2)^{-1}\xi_{2,h}]^2 \\ 1 \\ 1 \end{pmatrix}. \tag{2.18}$$

[8] Condition (3.2) in the definition of $\Gamma_3(\gamma_1, \gamma_2)$ ensures that we get the zero entries in the covariance matrix of the asymptotic distribution of $(\psi_{u,\rho}', \psi_{v,\rho}', \psi_{uv,\rho})$ and also that the right lower entry $(\sigma_u^{-2}\sigma_v^{-2})var(u_iv_i)$ in the covariance matrix equals $1+\rho^2$.

for $\eta'_h = (\eta_{1,h}, ..., \eta_{5,h})$.[9] Case II is dealt with in the Appendix. In Case II, the pretest statistic goes off to infinity, $H_n \rightarrow_p \infty$, and thus w.p.a.1, $T^*_{2SLS}(\theta_0)$ is used in the second stage. Because $T^*_{2SLS}(\theta_0) \rightarrow_d N(0,1)$, there is no size-distortion under the strong endogeneity of Case II.

We have

$$T^*_n(\theta_0) \rightarrow_d J^*_h, \tag{2.19}$$

where J^*_h, by definition, is the distribution of

$$\eta^*_h = \eta_{2,h} I(\eta_{3,h} \leq \chi^2_1(1-\beta)) + \eta_{1,h} I(\eta_{3,h} > \chi^2_1(1-\beta)). \tag{2.20}$$

The distribution J^*_h depends on the nominal size β of the pretest. For notational simplicity, this dependence is suppressed. The derivations above imply that Assumption B in AG (2005a) holds with $r = 1/2$.

Next some motivation is given for the size distortion of the two-stage tests. In the extreme case $h_2 = 0$ in Case I in (2.18), i.e. the unidentified case not allowed for in the above setup, the formulas in (2.18) read

$$\begin{aligned} T^*_{2SLS}(\theta_0) &\rightarrow_d s'_{k_2}\psi_{u,0}, \\ T^*_{OLS}(\theta_0) &\rightarrow_d \psi_{uv,0} + h_1, \text{ and} \\ H_n &\rightarrow_d (s'_{k_2}\psi_{u,0})^2 \sim \chi^2(1). \end{aligned} \tag{2.21}$$

It follows that in this situation, the Hausman pretest rejects with probability equal to β. When the Hausman test does not reject the pretest hypothesis (which happens with probability $1-\beta$) and thus the OLS based t-statistic is used in the second stage, the maximal asymptotic rejection probability for the null $H_0 : \theta = \theta_0$ equals 1. The latter is seen by picking h_1 very large or very negative depending on the type of test.[10] Picking a large nominal size β of the pretest, does not solve this problem. While picking a large β reduces the probability at which OLS based inference is performed in the second stage, it does not lower the conditional size of the second stage test, conditional on not rejecting the pretest null hypothesis. The potentially more powerful OLS based inference in the second stage comes at the price of extreme size distortion. If, for example, $\alpha = \beta = .05$, then the unconditional asymptotic size for the upper two-stage FCV test is at least 97.5%: With probability $1-\beta$, a t-statistic based on OLS is used and always rejects the null (for h_1 large enough) and with probability β, a t-statistic based on 2SLS is used which rejects the null

[9] Because $\eta_{3,h} = (1+h_2^2)^{-1}[s'_{k_2}\psi_{u,0} - h_2\psi_{uv,0} - h_2 h_1]^2$ and $s'_{k_2}\psi_{u,0} - h_2\psi_{uv,0} - h_2 h_1 \sim N(-h_2 h_1, 1+h_2^2)$, the limit distribution of H_n is $\chi^2_1(h_1^2 h_2^2 (h_2^2+1)^{-1})$. Therefore, $H_n \rightarrow_d \chi^2_1$ if $h_1 = 0$, that is under exogeneity and strong instruments, we obtain Hausman's (1978) result as a subcase. If $h_2 h_1 \neq 0$ the Hausman test has nonzero local power.

[10] Consider, for example, the case of an upper one-sided test. For every $\varepsilon > 0$ there exists a $h_1 = h_1(\varepsilon)$ such that $P(\psi_{uv,0} + h_1 > z_{1-\alpha}) > 1-\varepsilon$. Therefore, under the sequence $\rho_n = n^{-1/2}h_1$, asymptotically, conditional null rejection probabilities no smaller than $1-\varepsilon$ are obtained.

with probability 1/2. Intuitively, the pretest does not pick up the local invalidity of the exogeneity assumption, $\rho = n^{-1/2}h_1$. On the other hand, the mean of the limit distribution of the OLS based t-statistic is affected which leads to overrejection. By continuity, the same intuition applies for small values for h_2 rather than $h_2 = 0$. This is confirmed by the results below.

The next theorem gives an explicit formula for the asymptotic size $AsySz(\theta_0)$ of the two-stage test of $H_0 : \theta = \theta_0$ based on $T_n(\theta_0)$. The results apply to upper, lower one-sided, and symmetric two-sided versions of the test with η_h defined as η_h^*, $-\eta_h^*$, and $|\eta_h^*|$, respectively.

Theorem 1 *For upper, lower, and symmetric FCV tests based on* $T_n(\theta_0)$ *of nominal size* α*, the* $AsySz(\theta_0)$ *equals* $\sup_{h \in H} P(\eta_h > c_\infty(1-\alpha))$.

The proof follows from Theorem 1(a) in AG (2005a). Note that the asymptotic sizes depend on the pretest size β and on κ. For notational simplicity, this dependence is suppressed. Note that the results do not depend on k_1.

Table IIa contains information on the asymptotic size of the two-stage test when $k_2 = 5$ and $\alpha = .05$ for various values of κ and β, namely $\kappa \in \{.001, .1, .5, 1, 2, 10\}$ and $\beta \in \{.05, .1, .2, .5\}$.[11] Here and in the tables below, only results on upper and symmetric tests are reported. Results for lower (and equal-tailed) tests are virtually identical to the upper (and symmetric) ones. Note that a one-stage t-test based on the 2SLS estimator has asymptotic size equal to 5% whenever $\kappa > 0$.

Insert Table IIa about here

Naturally, $AsySz(\theta_0)$ is decreasing in both κ and β. Table IIa shows that $AsySz(\theta_0)$ by far exceeds the nominal size α for small numbers of κ and β. For example, when $\kappa = .1$ and $\beta = .05$ then the asymptotic size equals .93 and .95 for upper and symmetric tests, respectively. On the other hand, when $\kappa = 10$ and $\beta = .05$ then the asymptotic size equals .06 and .05 for upper and symmetric tests, respectively, and therefore basically equals the nominal size of the test. For $\beta = .05$ the symmetric test has asymptotic size equal to 1 for small lower bounds on the strength of the instrument.

To gain further insight, the asymptotic probability of the event "pretest does not reject the pretest null hypothesis" and the conditional probability of the event "test rejects the null hypothesis" conditional on the pretest not rejecting the pretest null hypothesis, are investigated. Table IIb contains the results for the case where $h_1 = 5$. For $\kappa \leq 1$, this conditional rejection probability is very close to or equal to 1 for

[11] In the simulations, $\overline{\kappa} = 1000$. Hansen, Newey, and Hausman (2004, Table 1) reports estimated concentration parameters μ^2 for the Angrist and Krueger (1991) data for two different setups with number of instruments equal to 3 and 180, respectively. The estimated concentration parameters are $\mu^2 = 95.6$ and 257, respectively. For the sample size $n = 329,509$ this implies $\gamma_2 = .017$ and .028, respectively.

both upper and symmetric tests for all nominal sizes β considered. Picking a large β decreases the asymptotic size of the two-stage test by more often using 2SLS based inference in the second stage, but it does not decrease the size problems of the test if OLS based inference is used in the second stage. The pretest does not detect a violation of the pretest null hypothesis, however the second stage t-statistic based on the OLS estimator takes on very large values. The probability $P(H_n < \chi_1^2(1-\beta))$ is of course decreasing in β and $\kappa \leq 1$. For $\beta = .05$ and $\kappa = .1$, it equals .92. The $AsySz(\theta_0)$ is large because, the pretest null hypothesis is not rejected with a large probability and conditional on this to happen, the second stage t-test based on OLS almost certainly rejects the null.

Insert Table IIb about here

3 Appendix

Definition of the set $\Gamma_3(\gamma_1,\gamma_2)$: Define

$$\begin{aligned}
&\Gamma_3(\gamma_1,\gamma_2)=\{(F,\pi,\zeta,\phi):\\
&E_F u_i=E_F v_i=0,\ E_F u_i^2=\sigma_u^2,\ E_F v_i^2=\sigma_v^2,\ E_F\overline{Z}_i\overline{Z}_i'=Q=\left[\begin{array}{cc} Q_{XX} & Q_{XZ}\\ Q_{ZX} & Q_{ZZ}\end{array}\right],\\
&\text{for some } \sigma_u^2,\sigma_v^2>0,\ \text{pd } Q\in R^{k\times k},\ \&\ \pi\in R^{k_2} \text{ that satisfy}\\
&Corr_F(u_i,v_i)=\gamma_1,\ ||\Omega^{1/2}\pi/\sigma_v||=\gamma_2 \text{ for } \Omega=Q_{ZZ}-Q_{ZX}Q_{XX}^{-1}Q_{XZ};\\
&\zeta,\phi\in R^{k_1};\ E_F u_i\overline{Z}_i=E_F v_i\overline{Z}_i=0;\ E_F(u_i^2,v_i^2,u_iv_i)\overline{Z}_i\overline{Z}_i'=(\sigma_u^2,\sigma_v^2,\sigma_u\sigma_v\rho)Q;\\
&E_F(u_i^2v_i\overline{Z}_i)=E_F(u_iv_i^2\overline{Z}_i)=0;\ var(u_iv_i)/(\sigma_u^2\sigma_v^2)=1+\gamma_1^2;\\
&\lambda_{\min}(E_F\overline{Z}_i\overline{Z}_i')\geq M^{-1};\ \left\|E_F\left(|u_i/\sigma_u|^{2+\delta},\ |v_i/\sigma_v|^{2+\delta},\ |u_iv_i/(\sigma_u\sigma_v)|^{2+\delta}\right)'\right\|\leq M,\ \&\\
&\left\|E_F\left(||\overline{Z}_iu_i/\sigma_u||^{2+\delta},\ ||\overline{Z}_iv_i/\sigma_v||^{2+\delta},\ ||\overline{Z}_i||^{2+\delta}\right)'\right\|\leq M\} \qquad (3.1)
\end{aligned}$$

for some constants $\delta>0$ and $M<\infty$, where “pd” denotes “positive definite.” The restrictions in $\Gamma_3(\gamma_1,\gamma_2)$ are similar to those in AG (2005c) and comprise exogeneity restrictions on $\overline{Z}$, moment restrictions that ensure the validity of central limit theorems and, for simplicity, conditional homoskedasticity is assumed. The additional conditions

$$E_F(u_i^2v_i\overline{Z}_i)=E_F(u_iv_i^2\overline{Z}_i)=0 \text{ and } var(u_iv_i)/(\sigma_u^2\sigma_v^2)=1+\rho^2, \qquad (3.2)$$

where $\rho=Corr_F(u_i,v_i)$, ensure that under exogeneity and strong instruments $\widehat{\theta}_{2SLS}-\widehat{\theta}_{OLS}$ is asymptotically uncorrelated with $\widehat{\theta}_{OLS}$. Hausman (1978) exploits the latter property when deriving the asymptotic variance of $\widehat{\theta}_{2SLS}-\widehat{\theta}_{OLS}$ when showing that $H\rightarrow_d\chi_1^2$ under strong instruments and exogeneity of y_2. Sufficient conditions for (3.2) are, for example, independence of (u_i,v_i) and $\overline{Z}_i$ and joint normality of (u_i,v_i) with zero mean.

Limit distribution of test statistic in Case II: Under sequences $\{\gamma_{n,h}\}$ for which $Corr_{F_n}(u_i,v_i)\rightarrow\rho$ and $h=(h_1,h_2)'$ with $|h_1|=\infty$ the following holds jointly

$$\left(\begin{array}{c} n^{-1/2}y_2'P_{Z^\perp}u/(\sigma_u\sigma_v)\\ n^{-1/2}[y_2'M_Xu-E_{F_n}u'v]/(\sigma_u\sigma_v)\\ n^{-1}y_2'P_{Z^\perp}y_2/\sigma_v^2\\ n^{-1}y_2'M_Xy_2/\sigma_v^2\end{array}\right)\rightarrow_d\xi_h=\left(\begin{array}{c} h_2s_{k_2}'\psi_{u,\rho}\\ h_2s_{k_2}'\psi_{u,\rho}+\psi_{uv,\rho}\\ h_2^2\\ h_2^2+1\end{array}\right) \qquad (3.3)$$

and

$$\left(\begin{array}{c} T_{2SLS}^*(\theta_0)\\ T_{OLS}^*(\theta_0)\\ H_n\\ \widehat{\sigma}_u^2(\widehat{\theta}_{2SLS})/\sigma_u^2\\ \widehat{\sigma}_u^2(\widehat{\theta}_{OLS})/\sigma_u^2\end{array}\right)\rightarrow_d\eta_h=\left(\begin{array}{c} s_{k_2}'\psi_{u,\rho}\\ h_1\\ \infty\\ 1\\ 1-\rho^2/(h_2^2+1)\end{array}\right). \qquad (3.4)$$

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TABLE Ia

Finite Sample Null Rejection Probabilities (in %) of Two-stage Test

$k_1 = 0$, $\alpha = \beta = .05$, $||\pi|| = \sqrt{23.6}n^{-1/2}$, $\rho = .279$; based on 50,000 repetitions

n	k_2	π_0	Upper	Sym	HPre	CondlUpper	CondlSym
100	1	.49	69.6	62.4	15.4	82.2	73.0
1000	1	.15	78.9	79.4	21.1	100	100
10000	1	.05	78.6	79.1	21.5	100	100
100	5	.22	70.9	63.2	14.0	82.4	73.0
1000	5	.07	80.7	81.0	19.3	100	100
10000	5	.02	80.3	80.7	19.7	100	100
100	20	.11	74.3	66.2	10.3	82.5	73.4
1000	20	.03	85.3	85.4	14.8	100	100
10000	20	.01	84.2	84.3	15.9	100	100

TABLE Ib

Finite Sample Null Rejection Probabilities (in %) of Symmetric Two-stage Test and 2SLS Based t-Test[12]

$k_1 = 0$, $\alpha = \beta = .05$, $n = 1000$, $k_2 = 5$; based on 50,000 repetitions

$\mu^2 \backslash \rho$	0	.05	.1	.2	.3	.4	.5	.6
0	5.1;0.0	34.9;0.0	88.5;0.1	99.9;0.4	99.9;2.4	99.9;8.5	99.9;22.2	99.9;42.6
13	6.7;0.7	35.2;0.8	86.8;1.3	95.4;3.1	91.0;5.9	83.8;10.0	71.6;14.9	53.8;20.6
50	7.8;3.4	34.5;3.5	81.4;3.6	77.1;4.2	50.0;5.3	21.2;6.7	8.0;8.4	7.7;10.2
113	7.8;4.4	32.3;4.4	74.0;4.4	51.6;4.7	15.3;5.1	5.5;5.7	5.7;6.6	6.3;7.5
200	7.4;4.8	29.7;4.7	65.1;4.8	29.2;4.9	5.8;5.0	5.2;5.4	5.3;5.8	5.7;6.4
313	7.1;4.9	27.1;4.9	55.4;4.9	15.1;4.9	5.1;5.1	5.1;5.3	5.3;5.5	5.4;5.9
450	6.8;5.0	24.6;5.0	46.3;4.9	8.8;5.0	5.1;5.1	5.1;5.2	5.2;5.4	5.3;5.6
613	6.5;5.0	22.2;5.0	38.4;5.0	6.5;5.0	5.1;5.1	5.1;5.2	5.2;5.3	5.2;5.4

[12] For each entry in the table, the first component is the finite sample null rejection probability of the two-stage test and the second component is the null rejection probability of the t-test based on 2SLS.

Table IIa[13]

$AsySz(\theta_0)$ (in %) of Two-stage FCV Test for $k_2 = 5$ and $\alpha = .05$

	Upper				Symmetric			
$\kappa\backslash\beta$	.05	.1	.2	.5	.05	.1	.2	.5
.001	97.4	94.8	84.9	55.4	100	94.9	85.0	55.6
.1	93.0	88.4	80.3	51.5	95.2	89.9	80.1	51.0
.5	62.4	52.9	40.6	23.1	58.6	50.0	38.9	21.4
1	30.0	24.2	18.5	10.5	27.0	20.4	15.8	9.9
2	13.5	11.1	8.8	6.5	10.7	9.3	7.7	6.2
10	5.9	5.6	5.4	5.2	5.3	5.3	5.2	5.2

Table IIb

Asymptotic Rejection Probabilities (in %) of Two-stage Test Conditional on Pretest Not Rejecting for $k_2 = 5$, $\alpha = .05$, $h_1 = 5$

	Upper				Symmetric				$P(H_n < \chi^2_1(1-\beta))$			
$\kappa\backslash\beta$	.05	.1	.2	.5	.05	.1	.2	.5	.05	.1	.2	.5
.001	100	100	100	100	99.9	99.9	99.9	99.9	94.9	89.8	79.8	50.3
.1	100	100	100	100	99.9	99.9	99.9	99.9	91.9	85.7	74.7	45.2
.5	99.7	99.8	99.8	99.8	99.4	99.4	99.5	99.5	39.1	27.5	16.9	5.7
1	97.5	97.4	97.3	96.9	94.5	94.8	95.2	94.8	5.7	2.9	1.1	0.2
2	71.7	71.2	71.4	74.5	60.8	59.8	60.3	61.4	0.5	0.2	0.1	0
10	13.0	14.8	14.8	14.3	8.5	10.0	9.0	8.6	0.1	0	0	0

[13] The results in Tables IIa and IIb are based on $R = 50,000$ simulation repetitions. If conditional events occur less than 100 times, the number of repetitions is increased.

Supplementary Appendix

Section 4 discusses power results of the two-stage test and a simple t-test based on 2SLS for the second experiment in Section 2.3. Section 5 discusses plug-in size-correction of the two-stage test for the application in Section 2 in the case where there is a positive lower bound on the strength of the instruments. The size-corrected version of the two-stage test is obtained by increasing the critical value of the test appropriately. The size-corrected critical value depends on the estimated strength of the instruments, using the plug-in methods introduced in AG (2005b). Section 6 derives the asymptotic size properties of the two-stage test for the application in Section 2 in a situation where weak instruments are allowed for. It is shown that then the asymptotic size equals 1 and that size-correction is no longer possible. Section 7 contains additional Monte Carlo evidence. Section 8 contains theoretical results on subsampling, hybrid (see AG (2005b)), and equal-tailed two-stage tests where a Hausman pretest is used in the first stage. It is shown that the subsampling versions of the two-stage test have asymptotic size equal to 1 and no size-correction is possible. Section 9 contains theoretical treatments of additional applications of Hausman pretests. In particular, in Subsection 9.1 the asymptotic size properties of a two-stage test are investigated when the second stage test-statistic is robust to weak instruments in the case when the Hausman pretest rejects the pretest null hypothesis of regressor exogeneity. The asymptotic size of this modified two-stage test is shown to equal 1. In Subsection 9.2, a Hausman pretest of instrument exogeneity is considered as in Hahn, Ham, and Moon (2007). Severe size-problems occur.

4 Power results

Table Ic, reports power results for the second experiment in Section 2.3. The null hypothesis is $H_0 : \theta = \theta_0 = 0$. The true value is $\theta = .1$ in the first chart of the table and $\theta = .2$ in the second chart of the table. The power of the two tests is virtually identical for the cases ($\rho \geq .3$ and $\mu^2 \geq 113$). If identification and endogeneity are large enough, the Hausman pretest rejects the pretest null hypothesis of exogeneity of the regressor, and in the second stage, inference based on 2SLS is used. The power gains of the two-stage procedure over the one-stage test for all other cases where $\rho > 0$ come at the price of size distortion of the two-stage test as documented above. If $\rho = 0$, the two-stage test is by far superior in terms of power and is not size-distorted in this case. Unfortunately, the researcher does not know whether $\rho = 0$ or whether $\rho > 0$ – this is why the pretest is implemented in the first place. But if $\rho > 0$, the two-stage procedure is often extremely size-distorted.

5 Plug-in Size Correction

In Section 2.5 it was shown that the two-stage test is size-distorted. The test can be size-corrected by increasing the critical value $c_\infty(1-\alpha)$ in (2.9) appropriately. In this section, following the work in AG (2005b), I discuss plug-in size-correction methods for the two-stage test that employ a consistent estimator $\widehat{\gamma}_{n,2}$ of the nuisance parameter $\gamma_{2,n} = ||(\Omega_n^{1/2}\pi_n/(E_{F_n}v_i^2)^{-1/2}||$. The idea is to use different critical values for different values of $\widehat{\gamma}_{n,2}$, rather than to use a critical value that is sufficiently large to work uniformly for all $\gamma_2 \in \Gamma_2$. This yields a more powerful test. Define the estimator

$$\begin{aligned}\widehat{\gamma}_{2,n} &= ||(\widehat{\Omega}_n^{1/2}\widehat{\pi}_n/\widehat{\sigma}_{v,n}|| \text{ for}\\ \widehat{\pi}_n &= (Z^{\perp\prime}Z^{\perp})^{-1}Z^{\perp\prime}y_2^{\perp},\\ \widehat{\Omega}_n &= n^{-1}Z^{\perp\prime}Z^{\perp}, \text{ and}\\ \widehat{\sigma}_{v,n}^2 &= n^{-1}(y_2^{\perp} - Z^{\perp\prime}\widehat{\pi}_n)'(y_2^{\perp} - Z^{\perp\prime}\widehat{\pi}_n)'. \end{aligned} \tag{5.5}$$

Under the technical assumption $\sigma_v^2 = o(n)$ it is easy to show that the estimator satisfies Assumption N of AG (2005b), namely, $\widehat{\gamma}_{n,2} - \gamma_{n,2} \to_p 0$ under all sequences $\{\gamma_n = (\gamma_{n,1}, \gamma_{n,2}, \gamma_{n,3}) \in \Gamma : n \geq 1\}$.

Denote by $c_h(1-\alpha)$ the $(1-\alpha)$-quantile of the distribution J_h^* in (2.19). Define

$$cv_{h_2}(1-\alpha) = \sup_{h_1 \in H_1} c_{(h_1,h_2)}(1-\alpha). \tag{5.6}$$

The plug-in size-corrected (PSC)-FCV two-stage test, rejects the null hypothesis if (2.9) holds with $c_\infty(1-\alpha)$ replaced by $cv_{\widehat{\gamma}_{2,n}}(1-\alpha)$.

The following theorem follows from Theorem 2 in AG (2005b).

Theorem 2 *If $\kappa > 0$ and $\sigma_v^2 = o(n)$ then the* PSC-FCV *test satisfies $AsySz(\theta_0) = \alpha$.*

6 The Weak IV Case

In this section, the asymptotic size properties of the two-stage test are discussed in a situation where weak instruments are no longer excluded. The weak instrument setup is interesting in the sense that there are several distinct sources of discontinuities in the limit distribution of the two-stage test. The first source is the correlation of the regressor and the structural error term, the second one is the potential weakness of the instruments, and the third one is an interaction term between the two. In all examples considered in AG (2005a-e) there is only one source of discontinuity. The asymptotic size of the two-stage test is 1. If instruments are potentially weak, size-correction of the two-stage test using the plug-in method is not possible.

6.1 Parameter Space

When the strength of the instruments, $||(\Omega^{1/2}\pi/\sigma_v||$, is not bounded away from zero, the nuisance parameter space Γ is much more complex. Assume again that $\{(u_i, v_i, X_i, Z_i) : i \leq n\}$ are i.i.d. with distribution F. Define the vector of nuisance parameters $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, $\gamma_1 = (\gamma_{11}, \gamma_{12}, \gamma_{13})$, $\gamma_2 = (\gamma_{21}, \gamma_{22})$ by

$$\begin{aligned}
&\gamma_1 = (||(\Omega^{1/2}\pi/\sigma_v||, \rho, \gamma_{11}\gamma_{12}) \in R^3,\ \gamma_2 = (\gamma_{11}, \gamma_{12}) \in R^2, \\
&\text{and } \gamma_3 = (F, \pi, \zeta, \phi),\ \text{where} \\
&\sigma_u^2 = E_F u_i^2,\ \sigma_v^2 = E_F v_i^2,\ \rho = Corr_F(u_i, v_i), \\
&\Omega = Q_{ZZ} - Q_{ZX}Q_{XX}^{-1}Q_{XZ},\ \text{and } Q = \begin{bmatrix} Q_{XX} & Q_{XZ} \\ Q_{ZX} & Q_{ZZ} \end{bmatrix} = E_F \overline{Z}_i \overline{Z}_i' \qquad (6.7)
\end{aligned}$$

and $||\cdot||$ denotes Euclidean norm. The first component of γ_1 measures the strength of the instruments and the second component the degree of endogeneity of y_2.[14] The third component is the product of the first two. If $n^{1/2}\gamma_{11} \to \infty$, $|n^{1/2}\gamma_{12}| \to \infty$, and $\gamma_2 \nrightarrow (0,0)$ then $n^{1/2}\gamma_{11}\gamma_{12} \to \lim n^{1/2}\gamma_{12}$ is pinned down. On the other hand, if $n^{1/2}\gamma_{11} \to \infty$, $|n^{1/2}\gamma_{12}| \to \infty$, and $\gamma_2 \to (0,0)$, the limit of $n^{1/2}\gamma_{11}\gamma_{12}$ could be any number in $sgn(\gamma_{12})R_{+,\infty}$. In that case, as shown in (6.25), the limit distribution of the Hausman statistic depends on the limit of $n^{1/2}\gamma_{11}\gamma_{12}$.

Note that $||(\Omega^{1/2}\pi/\sigma_v||$ and ρ appear in both vectors γ_1 and γ_2 because they influence the asymptotic distribution of $T_n(\theta_0)$ "continuously" and "discontinuously". Let $\Gamma_1 = \{\gamma_1 \in R^3; \{\gamma_{11}, \gamma_{12}\} \in [0, \overline{\kappa}] \times [-1, 1], \gamma_{13} = \gamma_{11}\gamma_{12}\}$ for some $\overline{\kappa} < \infty$.[15] For

[14] Note that in AG (2005a-e) the specification for γ has always been chosen such that when the components of γ times n^r diverge to infinity, we obtain the "standard FCV" asymptotic distribution. In this example, when $n^{1/2}|\gamma_{12}| \to \infty$, y_2 is not exogenous. Instead, the "standard" Hausman (1978) result $H_n \to_d \chi_1^2$ is obtained under $n^{1/2}|\gamma_{12}| \to 0$ and additional assumptions.

[15] Note that an upper bound $\overline{\kappa}$ is imposed on the component $\gamma_{11} = \gamma_{21}$ to avoid sequences γ_{21} that diverge to infinity. Allowing for such sequences would cause unnecessary complications in the asymptotic theory below. Removing the bound on $\overline{\kappa}$, the same asymptotic size results are obtained: The asymptotic size equals 1 with a bound on $\overline{\kappa}$ and therefore still equals 1 in the larger model where $\overline{\kappa}$ is unbounded.

given $\gamma_1 \in \Gamma_1$, define $\Gamma_2(\gamma_1) = \{(\gamma_{11}, \gamma_{12})\}$. Define

$$\begin{aligned}
&\Gamma_3(\gamma_1) = \{(F, \pi, \zeta, \phi) : \\
&\quad E_F u_i = E_F v_i = 0,\ E_F u_i^2 = \sigma_u^2,\ E_F v_i^2 = \sigma_v^2,\ E_F \overline{Z}_i \overline{Z}_i' = Q = \left[\begin{array}{cc} Q_{XX} & Q_{XZ} \\ Q_{ZX} & Q_{ZZ} \end{array}\right],\ \& \\
&\quad E_F u_i v_i / (\sigma_u \sigma_v) = \rho \text{ for some } \sigma_u^2, \sigma_v^2 > 0,\ \text{pd } Q \in R^{k \times k},\ \&\ \pi \in R^{k_2} \text{ that} \\
&\quad \text{satisfy } ||\Omega^{1/2} \pi / \sigma_v|| = \gamma_{11} \text{ for } \Omega = Q_{ZZ} - Q_{ZX} Q_{XX}^{-1} Q_{XZ},\ \rho = \gamma_{12}; \\
&\quad \zeta, \phi \in R^{k_1};\ E_F u_i \overline{Z}_i = E_F v_i \overline{Z}_i = 0;\ E_F (u_i^2, v_i^2, u_i v_i) \overline{Z}_i \overline{Z}_i' = (\sigma_u^2, \sigma_v^2, \sigma_u \sigma_v \rho) Q; \\
&\quad E_F (u_i^2 v_i \overline{Z}_i) = E_F (u_i v_i^2 \overline{Z}_i) = 0;\ var(u_i v_i)/(\sigma_u^2 \sigma_v^2) = 1 + \rho^2; \\
&\quad \lambda_{\min}(E_F \overline{Z}_i \overline{Z}_i') \geq M^{-1};\ \left\| E_F \left(|u_i/\sigma_u|^{2+\delta},\ |v_i/\sigma_v|^{2+\delta},\ |u_i v_i / (\sigma_u \sigma_v)|^{2+\delta} \right)' \right\| \leq M,\ \& \\
&\quad \left\| E_F \left(||\overline{Z}_i u_i / \sigma_u||^{2+\delta},\ ||\overline{Z}_i v_i / \sigma_v||^{2+\delta},\ ||\overline{Z}_i||^{2+\delta} \right)' \right\| \leq M\} \qquad (6.8)
\end{aligned}$$

for some constants $\delta > 0$ and $M < \infty$, where "pd" denotes "positive definite." The restrictions in $\Gamma_3(\gamma_1)$ are similar to those in AG (2005c) and comprise exogeneity restrictions on $\overline{Z}$, moment restrictions that ensure the validity of central limit theorems and, for simplicity, conditional homoskedasticity is assumed. The additional conditions

$$E_F(u_i^2 v_i \overline{Z}_i) = E_F(u_i v_i^2 \overline{Z}_i) = 0 \text{ and } var(u_i v_i)/(\sigma_u^2 \sigma_v^2) = 1 + \rho^2 \qquad (6.9)$$

ensure that under exogeneity and strong instruments $\widehat{\theta}_{2SLS} - \widehat{\theta}_{OLS}$ is asymptotically uncorrelated with $\widehat{\theta}_{OLS}$. Hausman (1978) exploits the latter property when deriving the asymptotic variance of $\widehat{\theta}_{2SLS} - \widehat{\theta}_{OLS}$ when showing that $H \rightarrow_d \chi_1^2$ under strong instruments and exogeneity of y_2. Sufficient conditions for (6.9) are, for example, independence of (u_i, v_i) and $\overline{Z}_i$ and joint normality of (u_i, v_i) with zero mean.

Finally, define the parameter space Γ as

$$\Gamma = \{\gamma = (\gamma_1, \gamma_2, \gamma_3) : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2(\gamma_1), \gamma_3 \in \Gamma_3(\gamma_1)\}. \qquad (6.10)$$

Unlike the definition of Γ in AG (2005a, eq. (5.1)), Γ does not have a product structure $\Gamma_1 \times \Gamma_2$ in the first two components (γ_1, γ_2) because the third component in γ_1 depends on the first two and $\Gamma_2 = \Gamma_2(\gamma_1)$ depends on γ_1.

6.2 Test Statistics and Critical Values

We use slightly different notation than before. Define the partially studentized t-test statistic

$$T_l^*(\theta) = \widehat{\sigma}_u(\widehat{\theta}_l) n^{1/2} (\widehat{\theta}_l - \theta) / \widehat{V}_l^{1/2} \qquad (6.11)$$

for $l = OLS$ and $2SLS$. Writing the test as in (6.11) using a partially studentized statistic, simplifies the asymptotic theory in situations where $\widehat{\sigma}_u$ converges to 0.

Also, for subsampling tests, studentizing is not necessary, see AG (2005c) for further discussion. Define the two-stage test statistic

$$T_n^*(\theta_0) = T_{OLS}^*(\theta_0)I(H_n \leq \chi_1^2(1-\beta)) + T_{2SLS}^*(\theta_0)I(H_n > \chi_1^2(1-\beta)), \qquad (6.12)$$

where, again, β is the nominal level of the pretest, I is the indicator function, and $\chi_1^2(1-\beta)$ the $1-\beta$ quantile of a chi-square random variable with one degree of freedom. Define the two-stage test statistic $T_n(\theta_0)$ as $\pm T_n^*(\theta_0)$ or $|T_n^*(\theta_0)|$ depending on whether the test is a lower/upper one-sided or a symmetric two-sided test, respectively.

The nominal $1-\alpha$ standard fixed critical value (FCV) test rejects H_0 if

$$\begin{aligned} &T_n(\theta_0) > c_\infty(1-\alpha)\widehat{\sigma}_u, \text{ where} \\ &\widehat{\sigma}_u = \widehat{\sigma}_u(\widehat{\theta}_{OLS})I(H_n \leq \chi_1^2(1-\beta)) + \widehat{\sigma}_u(\widehat{\theta}_{2SLS})I(H_n > \chi_1^2(1-\beta)), \end{aligned} \qquad (6.13)$$

$c_\infty(1-\alpha) = z_{1-\alpha}$, $z_{1-\alpha}$, and $z_{1-\alpha/2}$ for the upper one-sided, lower one-sided, and symmetric two-sided test, respectively and $z_{1-\alpha}$ is the $1-\alpha$ quantile of a standard normal distribution.

6.3 Asymptotic Distributions and Size

The tests above are equivalent to analogous tests defined with $T_l^*(\theta_0)$ and $\widehat{\sigma}_u$ replaced by

$$T_l^{**}(\theta_0) = T_l^*(\theta_0)/\sigma_u, \text{ and } \widehat{\sigma}_u/\sigma_u, \qquad (6.14)$$

respectively, where again $l = OLS$ or $2SLS$. Note that this also rescales $T_n^*(\theta_0)$ to $T_n^{**}(\theta_0) = T_n^*(\theta_0)/\sigma_{u}$. The reason for equivalence is that for all the tests above $1/\sigma_u$ scales both the test statistic and the critical value equally. In this subsection, the asymptotic distribution of the statistics written as in (6.14) are derived. This simplifies certain expressions in the asymptotic distributions that arise. Let $R_{+,\infty} = \{x \in R; x \geq 0\} \cup \{+\infty\}$ and $R_\infty = R \cup \{\pm\infty\}$. Let

$$\begin{aligned} H = \{&h = (h_1, h_2) \in R_\infty^{3+2} : \exists \{\gamma_n = (\gamma_{n,1}, \gamma_{n,2}, \gamma_{n,3}) \in \Gamma : n \geq 1\} \\ &\text{such that } n^{1/2}\gamma_{n,1} \to h_1 \text{ and } \gamma_{n,2} \to h_2\}. \end{aligned} \qquad (6.15)$$

Next an exact characterization of the set H is given. With $h_1 = (h_{11}, h_{12}, h_{13})$ and $h_2 = (h_{21}, h_{22})$ it follows that

$$H = \{h = (h_1, h_2);\ (h_{11}, h_{12}) \in R_{+,\infty} \times R_\infty,\ h_2 \in H_2(h_1), h_{13} \in H_{13}((h_{11}, h_{12}, h_2))\}, \qquad (6.16)$$

where $H_2(h_1) = H_{21}(h_{11}) \times H_{22}(h_{12})$,

$$H_{21}(h_{11}) = \begin{cases} \{0\} \text{ for } h_{11} < \infty \\ [0, \overline{\kappa}] \text{ for } h_{11} = \infty \end{cases}, \quad H_{22}(h_{12}) = \begin{cases} \{0\} \text{ for } |h_{12}| < \infty \\ [0,1] \text{ for } h_{12} = \infty \\ [-1, 0] \text{ for } h_{12} = -\infty \end{cases}, \qquad (6.17)$$

and

$$H_{13}((h_{11},h_{12},h_2)) = \begin{cases} \{0\} \text{ for } h_{11} < \infty \text{ and } |h_{12}| < \infty \\ \{h_{12}h_{21}\} \text{ for } h_{11} = \infty \text{ and } |h_{12}| < \infty \\ \{h_{11}h_{22}\} \text{ for } h_{11} < \infty \text{ and } |h_{12}| = \infty \\ sgn(h_{12})R_{+,\infty} \text{ for } h_{11} = |h_{12}| = \infty, h_{21} = h_{22} = 0 \\ \{h_{12}\} \text{ for } h_{11} = |h_{12}| = \infty, (h_{21} \neq 0 \text{ or } h_{22} \neq 0). \end{cases} \quad (6.18)$$

Note that except for the case $h_{11} = |h_{12}| = \infty$, ($h_{21} \neq 0$ or $h_{22} \neq 0$) the vector $(h_{11}, h_{12}, h_{21}, h_{22})$ uniquely pins down h_{13} and $H_{13}((h_{11}, h_{12}, h_2))$ is a singleton. Only in Case II, when $h_{21} = h_{22} = 0$, h_{13} is not uniquely pinned down and can take on any value in the set $sgn(h_{12})R_{+,\infty}$.

Let $h_1 = (h_{11}, h_{12}, h_{13})$ and $h_2 = (h_{21}, h_{22})$. There are four different cases. Case I has $h_{11} = \infty$ and $|h_{12}| < \infty$ (and consequently $h_{13} = h_{12}h_{21}$), Case II has $h_{11} = \infty$ and $|h_{12}| = \infty$, Case III has $h_{11} < \infty$ and $|h_{12}| = \infty$ (and thus $h_{13} = h_{11}h_{22}$), and Case IV has $h_{11} < \infty$ and $|h_{12}| < \infty$ (and thus $h_{13} = 0$). In Case II, when $h_{21} = h_{22} = 0$, h_{13} is not determined by the other components in h; in all other cases h_{13} is determined by the other components in h. In Cases I and IV, $h_{22} = 0$ and thus in the limit $var(u_i v_i)/(\sigma_u^2 \sigma_v^2) = 1$. In Cases III and IV $h_{21} = 0$. In Cases I and II, the instruments are strong while in Cases III and IV they are weak. In Cases I and IV y_2 is (essentially) exogenous while in Cases II and III it is endogenous.

Definition of $\{\gamma_{n,h}\}$: For $h = (h_1, h_2) \in H$, let $\{\gamma_{n,h}\} \subset \Gamma$ denote a sequence of parameters with components $\gamma_{n,h,1}, \gamma_{n,h,2}$, and $\gamma_{n,h,3}$, $\gamma_{n,h,1} = (\gamma_{n,h,1}, \gamma_{n,h,2}, \gamma_{n,h,3})'$, where

$$\begin{aligned} \gamma_{n,h,1} &= (||(\Omega_n^{1/2}\pi_n/(E_{F_n}v_i^2)^{-1/2}||, Corr_{F_n}(u_i, v_i), \gamma_{n,h,1}\gamma_{n,h,2}), \\ \Omega_n &= E_{F_n}Z_iZ_i' - E_{F_n}Z_iX_i'(E_{F_n}X_iX_i')^{-1}E_{F_n}X_iZ_i', \\ \gamma_{n,h,2} &= \gamma_{n,h,1}, \ n^{1/2}\gamma_{n,h,1} \to h_1, \ \gamma_{n,h,2} \to h_2, \text{ and} \\ \gamma_{n,h,3} &= (F_n, \pi_n, \zeta_n, \phi_n) \in \Gamma_3(\gamma_{n,h,1}). \end{aligned} \quad (6.19)$$

As Theorem 3 below shows, the highest asymptotic null rejection probability of the test is realized along some sequence $\{\gamma_{n,h}\}$. It is therefore enough to study the asymptotic rejection rates along sequences $\{\gamma_{n,h}\}$. Under any sequence $\{\gamma_{n,h}\}$, the following convergence result holds

$$\begin{aligned} &\begin{pmatrix} (n^{-1}Z^{\perp\prime}Z^{\perp})^{-1/2}n^{-1/2}Z^{\perp\prime}u/\sigma_u \\ (n^{-1}Z^{\perp\prime}Z^{\perp})^{-1/2}n^{-1/2}Z^{\perp\prime}v/\sigma_v \\ n^{-1/2}(u'v - E_{F_n}u'v)/(\sigma_u\sigma_v) \end{pmatrix} \to_d \begin{pmatrix} \psi_{u,h_{22}} \\ \psi_{v,h_{22}} \\ \psi_{uv,h_{22}} \end{pmatrix} \\ &\sim N(0, \begin{pmatrix} V_{h_{22}} \otimes I_{k_2} & 0 \\ 0' & 1 + h_{22}^2 \end{pmatrix}) \text{ for } V_{h_{22}} = \begin{bmatrix} 1 & h_{22} \\ h_{22} & 1 \end{bmatrix}, \end{aligned} \quad (6.20)$$

where $\psi_{u,h_{22}}, \psi_{v,h_{22}} \in R^{k_2}$, $\psi_{uv,h_{22}} \in R$. See AG (2005c, eq. (2.15)) for similar statements.[16]

Next the limit distribution of the test statistic $T_n^{**}(\theta_0)$ is derived under sequences $\gamma_{n,h}$. To do so, (6.20), (6.21), and derivations from AG (2005c, Sections 2.3 and 4.1.2) are used.

To derive the asymptotic theory, (6.20) and the following convergence results that hold jointly with (6.20), are used:

$$\begin{aligned}
&n^{-1}(u'u/\sigma_u^2, v'v/\sigma_v^2, u'v/(\sigma_u\sigma_v)) \rightarrow_p (1,1,h_{22}),\ n^{-1}\overline{Z}'[u{:}v] \rightarrow_p 0,\\
&\Omega_n^{-1}(n^{-1}Z^{\perp\prime}Z^{\perp}) \rightarrow_p I_{k_2},\ (E_{F_n}X_iX_i')^{-1}(n^{-1}X'X) \rightarrow_p I_{k_1}\ \&\\
&n^{-1}X'Z - E_{F_n}X_iZ_i' \rightarrow_p 0, \qquad (6.21)
\end{aligned}$$

see AG (2005c, (2.15) and (4.1)). The IV results of the following results are given in AG (2005c, (2.16)-(2.18)).

Straightforward but lengthy calculations using (6.20) and (6.21) below yield the following results for Case I; see AG (2005c) for similar statements. For $\xi_h = (\xi_{1,h},...,\xi_{4,h})'$, $h = (\infty, h_{12}, h_{12}h_{21}, h_{21}, 0)'$

$$\begin{pmatrix} n^{-1/2}y_2'P_{Z^\perp}u/(\sigma_u\sigma_v) \\ n^{-1/2}y_2'M_Xu/(\sigma_u\sigma_v) \\ n^{-1}y_2'P_{Z^\perp}y_2/\sigma_v^2 \\ n^{-1}y_2'M_Xy_2/\sigma_v^2 \end{pmatrix} \rightarrow_d \xi_h \sim \begin{pmatrix} h_{21}s_{k_2}'\psi_{u,0} \\ h_{21}s_{k_2}'\psi_{u,0} + \psi_{uv,0} + h_{12} \\ h_{21}^2 \\ h_{21}^2 + 1 \end{pmatrix}, \qquad (6.22)$$

where $s_{k_2} \in R^{k_2}$ is an arbitrary vector with $||s_{k_2}|| = 1$. Therefore, for $\eta_h = (\eta_{1,h},...,\eta_{5,h})'$

$$\begin{pmatrix} T_{2SLS}^{**}(\theta_0) \\ T_{OLS}^{**}(\theta_0) \\ H_n \\ \widehat{\sigma}_u^2(\widehat{\theta}_{2SLS})/\sigma_u^2 \\ \widehat{\sigma}_u^2(\widehat{\theta}_{OLS})/\sigma_u^2 \end{pmatrix} \rightarrow_d \eta_h = \begin{pmatrix} s_{k_2}'\psi_{u,0} \\ (1+h_{21}^2)^{-1/2}\xi_{2,h} \\ (1+h_{21}^2)[s_{k_2}'\psi_{u,0} - h_{21}(1+h_{21}^2)^{-1}\xi_{2,h}]^2 \\ 1 \\ 1 \end{pmatrix} \qquad (6.23)$$

by using (6.26).[17]

In Case II, jointly under $\{\gamma_{n,h}\}$ for $h = (\infty, h_{12}, h_{13}, h_{21}, h_{22})'$ with $|h_{12}| = \infty$

$$\begin{pmatrix} n^{-1/2}y_2'P_{Z^\perp}u/(\sigma_u\sigma_v) \\ n^{-1/2}[y_2'M_Xu - E_{F_n}u'v]/(\sigma_u\sigma_v) \\ n^{-1}y_2'P_{Z^\perp}y_2/\sigma_v^2 \\ n^{-1}y_2'M_Xy_2/\sigma_v^2 \end{pmatrix} \rightarrow_d \xi_h = \begin{pmatrix} h_{21}s_{k_2}'\psi_{u,h_{22}} \\ h_{21}s_{k_2}'\psi_{u,h_{22}} + \psi_{uv,h_{22}} \\ h_{21}^2 \\ h_{21}^2 + 1 \end{pmatrix}. \qquad (6.24)$$

[16] Condition (6.9) in the definition of $\Gamma_3(\gamma_1)$ ensures that we get the zero entries in the covariance matrix of the asymptotic distribution of $(\psi_{u,h_{22}}', \psi_{v,h_{22}}', \psi_{uv,h_{22}})$ and also that the (3,3) entry $(\sigma_u^{-2}\sigma_v^{-2})var(u_iv_i)$ in the matrix equals $1 + h_{22}^2$.

[17] Because $\eta_{3,h} = (1+h_{21}^2)^{-1}[s_{k_2}'\psi_{u,0} - h_{21}\psi_{uv,0} - h_{21}h_{12}]^2$ and $s_{k_2}'\psi_{u,0} - h_{21}\psi_{uv,0} - h_{21}h_{12} \sim N(-h_{21}h_{12}, 1+h_{21}^2)$, the limit distribution of H_n is $\chi_1^2(h_{12}^2h_{21}^2(h_{21}^2+1)^{-1})$. Therefore, $H_n \rightarrow_d \chi_1^2$ if $h_{12} = 0$, that is under exogeneity and strong instruments, we obtain Hausman's (1978) result as a subcase. If $h_{21}h_{12} \neq 0$ the Hausman test has local power.

Therefore, as shown below, for $\eta_h = (\eta_{1,h}, ..., \eta_{5,h})'$, we have

$$\begin{pmatrix} T^{**}_{2SLS}(\theta_0) \\ T^{**}_{OLS}(\theta_0) \\ H_n \\ \widehat{\sigma}^2_u(\widehat{\theta}_{2SLS})/\sigma^2_u \\ \widehat{\sigma}^2_u(\widehat{\theta}_{OLS})/\sigma^2_u \end{pmatrix} \rightarrow_d \eta_h = \begin{pmatrix} s'_{k_2}\psi_{u,h_{22}} \\ h_{12} \\ (s'_{k_2}\psi_{u,0} - h_{13})^2 \\ 1 \\ 1 - h^2_{22}/(h^2_{21}+1) \end{pmatrix} \tag{6.25}$$

and again $T_{OLS}(\theta_0)$ goes off to plus or minus infinity.

Note that if $h_{13} = \infty$ then H_n goes off to infinity and asymptotically $T^{**}_n(\theta_0) = T^{**}_{2SLS}(\theta_0)$ with probability 1. On the other hand, if $h = (\infty, \infty, 0, 0, 0)'$ it follows that $\widehat{\sigma}^2_u(\widehat{\theta}_{OLS})/\sigma^2_u \rightarrow_p 1$, $\widehat{\sigma}^2_u(\widehat{\theta}_{2SLS})/\sigma^2_u \rightarrow_p 1$, and $H_n \rightarrow_d (s'_{k_2}\psi_{u,0})^2 \sim \chi^2(1)$. In particular, the pretest rejects with probability β. If it does not reject, which happens with probability $1-\beta$, the second stage test (for the upper and symmetric case) rejects with probability 1 because $T^{**}_{OLS}(\theta_0) \rightarrow \infty$ (and analogously for lower type tests by looking at the case $h = (\infty, -\infty, 0, 0, 0)'$). If the pretest rejects, then the second stage test rejects with probability 50% for upper and lower tests and with 100% for a symmetric test.

To see that $H_n \rightarrow_d (s'_{k_2}\psi_{u,0} - h_{13})^2$ in (6.25), note that from

$$H_n = \frac{\frac{\sigma^2_v}{\sigma^2_u}\left(\frac{y'_2 P_{Z^\perp} u}{y'_2 P_{Z^\perp} y_2} - \frac{y'_2 M_X u}{y'_2 M_X y_2}\right)^2}{\frac{\widehat{\sigma}^2_u(\widehat{\theta}_{2SLS})}{\sigma^2_u}\frac{\sigma^2_v}{y'_2 P_{Z^\perp} y_2} - \frac{\widehat{\sigma}^2_u(\widehat{\theta}_{OLS})}{\sigma^2_u}\frac{\sigma^2_v}{y'_2 M_X y_2}}, \tag{6.26}$$

it follows that up to lower order terms

$$\begin{aligned} H_n &= n\frac{[n^{-1/2}\gamma^{-1}_{n,h,2,1}s'_{k_2}\psi_{u,h_{22}} - \gamma_{n,h,2,2}(h^2_{21}+1)^{-1}]^2}{\gamma^{-2}_{n,h,2,1} - (1 - \frac{h^2_{22}}{h^2_{21}+1})\frac{1}{h^2_{21}+1}} \\ &= \frac{[s'_{k_2}\psi_{u,h_{22}} - n^{1/2}\gamma_{n,h,2,1}\gamma_{n,h,2,2}(h^2_{21}+1)^{-1}]^2}{(h^2_{21}+1)^{-2}(1 + h^2_{21} + h^2_{21}h^2_{22})}, \end{aligned} \tag{6.27}$$

where $\gamma_{n,h,2,i}$ denotes the i-th element of $\gamma_{n,h,2}$ for $i = 1, 2$. The denominator in the second line of (6.27) is always a positive number. If $h_{21} \neq 0$ or $h_{22} \neq 0$ (which implies $|n^{1/2}\gamma_{n,h,2,1}\gamma_{n,h,2,2}| \rightarrow \infty$) then $H_n \rightarrow_d \infty$ follows. If $h_{21} = h_{22} = 0$, then up to lower order terms $H_n = (s'_{k_2}\psi_{u,0} - n^{1/2}\gamma_{n,h,2,1}\gamma_{n,h,2,2})^2$.

In Case III, we have jointly under $\{\gamma_{n,h}\}$ for $h = (h_{11}, h_{12}, h_{11}h_{22}, 0, h_{22})'$ with $|h_{12}| = \infty$,

$$\begin{pmatrix} y'_2 P_{Z^\perp} u/(\sigma_u\sigma_v) \\ n^{-1/2}[y'_2 M_X u - E_{F_n}u'v]/(\sigma_u\sigma_v) \\ y'_2 P_{Z^\perp} y_2/\sigma^2_v \\ n^{-1}y'_2 M_X y_2/\sigma^2_v \end{pmatrix} \rightarrow_d \xi_h = \begin{pmatrix} (\psi_{v,h_{22}} + h_{11}s_{k_2})'\psi_{u,h_{22}} \\ \psi_{uv,h_{22}} \\ (\psi_{v,h_{22}} + h_{11}s_{k_2})'(\psi_{v,h_{22}} + h_{11}s_{k_2}) \\ 1 \end{pmatrix} \tag{6.28}$$

and therefore,

$$\begin{pmatrix} T^{**}_{2SLS}(\theta_0) \\ T^{**}_{OLS}(\theta_0) \\ H_n \\ \widehat{\sigma}^2_u(\widehat{\theta}_{2SLS})/\sigma^2_u \\ \widehat{\sigma}^2_u(\widehat{\theta}_{OLS})/\sigma^2_u \end{pmatrix} \rightarrow_d \eta_h = \begin{pmatrix} \xi_{1,h}/\xi^{1/2}_{3,h} \\ h_{12} \\ ((\xi_{1,h}/\xi_{3,h}) - h_{22})^2/(\xi_{5,h}/\xi_{3,h}) \\ (1 - h_{22}\xi_{1,h}/\xi_{3,h})^2 + (1 - h^2_{22})\xi^2_{1,h}/\xi^2_{3,h} \\ 1 - h^2_{22} \end{pmatrix}. \tag{6.29}$$

In this case, $T^{**}_{OLS}(\theta_0)$ goes off to plus or minus infinity. Note that $\psi_{uv,h_{22}} \sim N(0, 1 + h^2_{22})$.

In Case IV, we have jointly under $\{\gamma_{n,h}\}$ for $h = (h_{11}, h_{12}, 0, 0, 0)'$,

$$\begin{pmatrix} y_2' P_{Z^\perp} u/(\sigma_u \sigma_v) \\ n^{-1/2} y_2' M_X u/(\sigma_u \sigma_v) \\ y_2' P_{Z^\perp} y_2/\sigma^2_v \\ n^{-1} y_2' M_X y_2/\sigma^2_v \end{pmatrix} \rightarrow_d \xi_h = \begin{pmatrix} (\psi_{v,0} + h_{11} s_{k_2})' \psi_{u,0} \\ h_{12} + \psi_{uv,0} \\ (\psi_{v,0} + h_{11} s_{k_2})'(\psi_{v,0} + h_{11} s_{k_2}) \\ 1 \end{pmatrix}, \tag{6.30}$$

where s_{k_2} is any vector in R^{k_2} with $||s_{k_2}|| = 1$. Therefore, we have

$$\begin{pmatrix} T^{**}_{2SLS}(\theta_0) \\ T^{**}_{OLS}(\theta_0) \\ H_n \\ \widehat{\sigma}^2_u(\widehat{\theta}_{2SLS})/\sigma^2_u \\ \widehat{\sigma}^2_u(\widehat{\theta}_{OLS})/\sigma^2_u \end{pmatrix} \rightarrow_d \eta_h = \begin{pmatrix} \xi_{1,h}/\xi^{1/2}_{3,h} \\ h_{12} + \psi_{uv,0} \\ (\xi^2_{1,h}/\xi_{3,h})/(1 + (\xi_{1,h}/\xi_{3,h})^2) \\ 1 + \xi^2_{1,h}/\xi^2_{3,h} \\ 1 \end{pmatrix}. \tag{6.31}$$

Note that asymptotically the OLS-components of H_n are dominated by the 2SLS-components and do not appear in the asymptotic distribution of H_n.

In all Cases I-IV we then have

$$T^{**}_n(\theta_0) \rightarrow_d J^{**}_h, \tag{6.32}$$

where J^{**}_h, by definition, is the distribution of

$$\eta^{**}_h = \eta_{2,h} I(\eta_{3,h} \leq \chi^2_1(1-\beta)) + \eta_{1,h} I(\eta_{3,h} > \chi^2_1(1-\beta)) \tag{6.33}$$

and

$$\widehat{\sigma}_u/\sigma_u \rightarrow_d J_{u,h}, \tag{6.34}$$

where $J_{u,h}$, by definition, is the distribution of

$$\eta_{u,h} = \eta^{1/2}_{5,h} I(\eta_{3,h} \leq \chi^2_1(1-\beta)) + \eta^{1/2}_{4,h} I(\eta_{3,h} > \chi^2_1(1-\beta)). \tag{6.35}$$

The distribution $J_{u,h}$ depends on the pretest nominal size β. For notational simplicity, this dependence is suppressed. The derivations above imply that Assumption B in AG (2005a) holds with $r = 1/2$.

The motivation for the size distortion of the two-stage tests is fully analogous to the discussion in Subsection 2.5. The next theorem gives an explicit formula for the asymptotic size $AsySz(\theta_0)$ of the two-stage test of $H_0 : \theta = \theta_0$ based on $T_n(\theta_0)$ and FCV. The results apply to upper, lower one-sided, and symmetric two-sided versions of the test with η_h defined as η_h^{**}, $-\eta_h^{**}$, and $|\eta_h^{**}|$, respectively.

Theorem 3 *For upper, lower, and symmetric FCV tests based on $T_n(\theta_0)$ of nominal size α, the $AsySz(\theta_0)$ equals $\sup_{h \in H} P(\eta_h > \eta_{u,h} c_\infty(1-\alpha))$.*

Proof. A straightforward modification of Theorem 3 in AG (2005d) from the asymptotic confidence size of confidence intervals to the asymptotic size of tests gives the desired result, noting that Assumptions A0 and B0 in AG (2005d) hold. Note that Theorem 1 in AG (2005a) can not be applied here because the parameter space does not have a product structure. Assumption A0 holds trivially and Assumption B0 holds because the result (6.32) that has been verified under sequences $\{\gamma_{n,h}\}$ also holds under subsequences w_n of n. □

Note that the asymptotic sizes depend on the pretest size β. For notational simplicity, this dependence is suppressed. Note that the results do not depend on k_1. For $\alpha = \beta = .05$ and $k_2 = 5$, evaluation of the formulas imply that $AsySz = 1$ for all versions of the two-stage tests considered. As argued above, the conditional size of the tests, conditional on the Hausman pretest not rejecting, is 1.

Table IIc contains information on the asymptotic size of the above tests when $k_2 = 5$ and $\alpha = \beta = .05$. Here and in the tables below, only results on upper and symmetric tests are reported. Results for lower and equal-tailed tests are virtually identical to the upper and symmetric ones, respectively. Table IIc also contains results on the maximum asymptotic null rejection probabilities of the two-stage test when the maximum is only taken over h values that conform with the restrictions imposed by Cases I-IV. Table IIc shows that asymptotic null rejection probabilities equal to 1 occur in a wide array of parameter combinations that include the cases of weak and strong instruments and the case of weak and strong endogeneity of the regressor y_2.

Table IIc[18]

Maximal Null Rejection Probabilities of Two-stage FCV Test for Cases I-IV and $AsySz(\theta_0)$ for $k_2 = 5$ and $\alpha = \beta = .05$

Type\Case	I	II	III	IV	$AsySz(\theta_0)$
Upper	97.4	97.5	100	99.9	100
Symmetric	100	100	100	100	100

[18]The results in this table are based on $R = 50,000$ simulation repetitions. In columns 2-5, the maximum null rejection probabilities are given for upper and symmetric tests, where the maximum is taken over a fine grid of h vectors that satisfy the restrictions imposed by the particular Case considered. For example, Case I has $h_{11} = \infty$ and h_{12} finite. The maximum is taken over a fine grid of $h_{12} \in R$ and $h_{21} \in [0, \overline{\kappa}]$ values. The constant $\overline{\kappa}$ is taken as 1000.

7 Finite Sample Results

In this section, the results of a small Monte Carlo study are reported that reflect the asymptotic results from Sections 2 and 6. The study shows that size distortion of the two-stage test occurs for a wide array of parameter combinations by modelling sequences of parameter values that fall into each of the four cases, Case j for $j \in \{I, ..., IV\}$, considered above.

The model considered is intentionally simple. The asymptotic results do not depend on k_1, the number of included exogenous variables, and therefore I take $k_1 = 0$. I also take only one instrument $k_2 = 1$. The nominal sizes of the pretest and the second stage test are $\alpha = \beta = .05$. The vector (u_i, v_i, Z_i) is i.i.d. normal with zero mean and unit variances and Z_i is independent of u_i and v_i.

The parameters $Corr(u_i, v_i) = \rho$ and π are modelled as functions of the sample size n. More precisely, for Case I, let $\rho = 10n^{-1/2}; \pi = n^{-1/4}$. For Case II, consider $\rho = n^{-1/4}; \pi = n^{-1/4}$, for Case III, let $\rho = n^{-1/4}; \pi = 10n^{-1/2}$, and for Case IV, let $\rho = 10n^{-1/2}; \pi = 10n^{-1/2}$. Two additional cases, Case I' and Case II' will be defined below. Various values of n are considered, namely $n \in \{100, 1000, 10000, 100000\}$.

Table Id provides finite sample null rejection probabilities of the two-stage test, rejection probabilities of the pretest, and conditional rejection probabilities of the two-stage test conditional on the pretest not rejecting the pretest null hypothesis. Only results for upper and symmetric tests are reported because the lower and equal-tailed results are fully analogous. The same column headings as in Table Id are used, namely "Upper" and "Sym" for the finite sample null rejection probabilities for upper and symmetric two-stage tests, "HPre" for the pretest null rejection probabilities, and "CondlUpper" and "CondlSym" for the conditional rejection probabilities. The charts I-IV, I', and IV' state the results for Cases I-IV described above and Cases I' and II' described below.

Insert Table Id about here

The simulation results reflect well the asymptotic findings and show that extreme size distortion of the two-stage test is a common situation that covers situations in which the instrument is weak or strong and endogeneity is weak or strong. The conditional results seem to indicate that the main cause of size distortion is the failure of the Hausman pretest to reject the pretest null hypothesis in situations where the pretest null is "locally" violated. In these situations, the resulting second stage t-statistic based on the OLS estimator rejects with very high probabilities. Actually, the conditional rejection probability equals 100% for all cases considered with $n \geq 1000$.

If the Case II setup is modelled as $\rho = n^{-1/3.5}$ and $\pi = n^{-1/3.5}$ (instead of $\rho = n^{-1/4}$ and $\pi = n^{-1/4}$), a case not reported in Table Id, then $h_{13} = 0$ and the discussion below (6.25) predicts that the problem of overrejection becomes more severe. Indeed, for $n = 100000$, $\rho = .04$, and $\pi = .04$ the simulated null rejection probabilities for the upper and symmetric two-stage tests are 94.0% and 96.4%, respectively.

There are however situations where the overrejection of the two-stage test is not as severe or where there is no overrejection at all. For example, in a Case I situation, where $\rho = 10n^{-1/2}$ and $\pi = 1$ identification is strong and endogeneity is weak. Because $H_n \rightarrow_d \chi_1^2(h_{12}^2 h_{21}^2 (h_{21}^2 + 1)^{-1})$ in this situation, the Hausman pretest has power against the local alternative and with high percentage a t-statistic based on 2SLS is used in the second stage. See chart I' in Table Id. Also, in a Case II situation where $\rho = .2$ and $\pi = 1$, it follows that $h_{13} = \infty$ and $H_n \rightarrow \infty$. Thus, asymptotically in the second stage a t-statistic based on the 2SLS is used with probability 1 that rejects with probability α. Chart II' in Table Id shows that the finite sample null rejection probabilities are very close to 5% in this situation. In the setup of charts I' and II', where the two-stage procedure has correct null rejection probabilities, a one-stage procedure based on a 2SLS t-statistic would of course also have correct null rejection probabilities. In addition, the one-stage procedure would have correct null rejection probabilities in charts I and II, whereas the two-stage procedure is extremely size distorted.

8 Subsampling, Hybrid, and Equal-Tailed Tests

This section contains theoretical results on subsampling, hybrid (see AG (2005b)), and equal-tailed two-stage tests where a Hausman pretest is used in the first stage in the case where weak instruments are allowed for. The asymptotic size of the subsampling versions of the two-stage test is 1. A priori, it is not clear that subsampling versions of the two-stage test have asymptotic size equal to 1 in weak instrument scenarios. Note that, for example, in the linear IV model a two-sided symmetric confidence interval based on inverting a t-statistic using normal FCVs has asymptotic confidence size equal to zero, but has virtually correct asymptotic confidence size for subsampling critical values, see Dufour (1997) and AG (2005c) for details.

In this section, subsampling and hybrid critical values are defined. The latter are introduced and discussed in AG (2005b). Also, critical values for equal-tailed FCV, subsampling, and hybrid tests are discussed.

For subsampling and hybrid tests, let b denote the subsample size, which depends on n. The number of different subsamples of size b is denoted by q_n. With i.i.d. observations, there are $q_n = n!/((n-b)!b!)$ different subsamples of size b. Let $\{T_{n,b,j}^*(\theta_0) : j = 1, ..., q_n\}$ be subsample statistics that are defined exactly as $T_n^*(\theta_0)$ in (6.12) is defined, but are based on subsamples of the data of size b rather than the full sample. Define $\{T_{n,b,j}(\theta_0) : j = 1, ..., q_n\}$ in the obvious way. The blocksize b satisfies $b \rightarrow \infty$ and $b/n \rightarrow 0$. The empirical distribution of $\{T_{n,b,j}(\theta_0) : j = 1, ..., q_n\}$ is

$$U_{n,b}(x) = q_n^{-1} \sum_{j=1}^{q_n} 1(T_{n,b,j}(\theta_0) \leq x). \tag{8.36}$$

The nominal $1 - \alpha$ upper and lower one-sided and symmetric two-sided subsample tests reject H_0 if $T_n(\theta_0) > c_{n,b}(1-\alpha)$, where $c_{n,b}(1-\alpha)$ is the $1-\alpha$ quantile of $U_{n,b}(x)$.[19]

The nominal $1-\alpha$ hybrid test is defined to reject H_0 if

$$T_n(\theta_0) > \max\{c_{n,b}(1-\alpha), c_\infty(1-\alpha)\widehat{\sigma}_u\}. \tag{8.37}$$

Finally, equal-tailed tests are defined. A nominal level α ($\in (0,1/2)$) equal-tailed t-test rejects H_0 when

$$T_n(\theta_0) > c_{1-\alpha/2} \text{ or } T_n(\theta_0) < c_{\alpha/2}, \tag{8.38}$$

where $c_{1-\alpha} = c_\infty(1-\alpha)$ for FCV tests, $c_{1-\alpha} = c_{n,b}(1-\alpha)$ for subsampling tests and $c_{1-\alpha/2} = \max\{c_{n,b}(1-\alpha/2), c_\infty(1-\alpha/2)\}$ and $c_{\alpha/2} = \min\{c_{n,b}(\alpha/2), c_\infty(\alpha/2)\}$ for the hybrid test. The exact size, $ExSz_n(\theta_0)$, of the equal-tailed t-test is

$$ExSz_n(\theta_0) = \sup_{\gamma\in\Gamma}\left(P_{\theta_0,\gamma}(T_n(\theta_0) > c_{1-\alpha/2}) + P_{\theta_0,\gamma}(T_n(\theta_0) < c_{\alpha/2})\right). \tag{8.39}$$

The asymptotic size of the test is then again $AsySz(\theta_0) = \limsup_{n\to\infty} ExSz_n(\theta_0)$.

Similar to AG (2005d), define

Definition of $\{\gamma_{w_n,g,h} : n \geq 1\}$: For $g = (g_1, g_2) \in R_\infty^5$, and $h = (h_1, h_2) \in R_\infty^5$ with $g_2 = h_2$, let $\{\gamma_{w_n,g,h} = (\gamma_{w_n,g,h,1}, \gamma_{w_n,g,h,2}, \gamma_{w_n,g,h,3}) : n \geq 1\}$ denote a sequence of parameters in Γ for which $w_n^{1/2}\gamma_{w_n,g,h,1} \to h_1$, $b_{w_n}^{1/2}\gamma_{w_n,g,h,1} \to g_1$, $\gamma_{w_n,g,h,2} \to h_2$, if such a sequence exists.

By definition, a sequence $\{\gamma_{w_n,g,h} : n \geq 1\}$ also is of the form $\{\gamma_{w_n,h} : n \geq 1\}$.

The index set of the asymptotic distributions of $T_{w_n}(\theta_0)$ and $T_{w_n,b_{w_n},j}(\theta_0)$ under sequences $\{\gamma_{w_n,g,h} : n \geq 1\}$ is denoted by GH. By definition,

$$\begin{aligned} GH = \{(g,h) \in R_\infty^5 \times R_\infty^5 : \exists \text{ a subsequence } \{w_n\} \text{ and} \\ \text{a sequence } \{\gamma_{w_n,g,h} : n \geq 1\}\}. \end{aligned} \tag{8.40}$$

Note that the set GH may depend on the relative size of b with respect to n. In fact, it does here, see below.

Let $c_h(1-\alpha)$ be the $1-\alpha$ quantile of J_h, where J_h is defined as $J_h^{**}, -J_h^{**}$, and $|J_h^{**}|$, respectively, and J_h^{**} is defined on top of 6.33. The next theorem gives formulas for the asymptotic sizes of subsampling and hybrid two-stage tests of $H_0 : \theta = \theta_0$.

[19] The subsample statistics are evaluated at the null value θ_0 and, hence, satisfy Assumption Sub2 of AG (2005a). Evaluating them at $\widehat{\theta}_n$ is generally not recommended because $\widehat{\theta}_n$ is not a consistent estimator of θ_0 when the IVs are weak. See Guggenberger and Wolf (2004). Under Assumption Sub2, Assumption G2 in AG (2005a) holds trivially.

Theorem 4 *For upper, lower, and symmetric tests based on $T_n(\theta_0)$ of nominal size α, the subsampling and hybrid test have asymptotic sizes $AsySz(\theta_0)$ equal to*

$$\sup_{(g,h)\in GH} [1 - J_h(c_g(1-\alpha))],$$
$$\sup_{(g,h)\in GH} P(\eta_h > \max\{c_g(1-\alpha), \eta_{u,h}c_\infty(1-\alpha)\}),$$

respectively. The asymptotic sizes $AsySz(\theta_0)$ for subsampling, FCV, and hybrid equal-tailed test equal

$$\sup_{(g,h)\in GH} [1 - J_h(c_g(1-\alpha/2)) + J_h(c_g(\alpha/2))],$$
$$\sup_{h\in H} P(\eta_h > \eta_{u,h}c_\infty(1-\alpha/2) \text{ or } \eta_h < \eta_{u,h}c_\infty(\alpha/2)),$$
$$\sup_{(g,h)\in GH} P(\eta_h > \max\{c_g(1-\alpha/2), \eta_{u,h}c_\infty(1-\alpha/2)\} \text{ or } \eta_h < \min\{c_g(\alpha/2), \eta_{u,h}c_\infty(\alpha/2)\}).$$

Again for notational simplicity, the possible dependence on β is suppressed in the notation. The proof of the above theorem follows as a straightforward modification of Theorem 3 in AG (2005d).

The set GH can be written as $\cup_{j\in J} GH_j$ for $j \in \{I, II, III, IV\}$, where GH_j is the subset of elements (g,h) in GH for which h satisfies the restriction of Case j. For $j = I$, $h_{11} = \infty$, $|h_{12}| < \infty$, and

$$\begin{aligned} GH_I = \{&(\infty, h_{12}, h_{12}h_{21}, h_{21}, 0) \times (\infty, 0, 0, h_{21}, 0); h_{12} \in R \ \&\ h_{21} \in (0, \overline{\kappa}]\} \\ \cup\{&(\infty, h_{12}, 0, 0, 0) \times (h_{11}, 0, 0, 0, 0); h_{12} \in R \ \&\ h_{11} \in R_{+,\infty}\}. \end{aligned} \quad (8.41)$$

The characterization of GH_{II} depends on the relative sizes of b and n. For example, let $w_n^{1/2}\gamma_{w_n,g,h,1} \to h_1 = (\infty, \infty, h_{13})$ for $0 < h_{13} < \infty$ and $h = (h_1, (0,0))$. In particular, $w_n^{1/2}\gamma_{w_n,g,h,1,1}\gamma_{w_n,g,h,1,2} \to h_{13}$, which implies that $\gamma_{w_n,g,h,1,1}$ or $\gamma_{w_n,g,h,1,2}$ is at most of order $w_n^{-1/4}$. If $b_{w_n} = w_n^{1/3}$ then for $(g,h) \in GH$ with $h_1 = (\infty, \infty, h_{13})$ it is possible that $(g_{11}, g_{12}) = (\infty, \infty)$; on the other hand, if $b_{w_n} = w_n^{1/5}$ then necessarily, one component of (g_{11}, g_{12}) equals 0. One could characterize GH for various choices of b but this is not necessary here. For the purpose here, it is enough to find a subset of GH_{II} whose definition is not affected by the particular choice of b, and on which the size of the two-stage test is 1. The set

$$\begin{aligned} \widetilde{GH}_{II} = \{&(\infty, h_{12}, h_{12}, h_{21}, h_{22})^2; |h_{12}| = \infty,\ h_{21} \in (0, \overline{\kappa}],\ \&\ h_{22} \in sgn(h_{12})(0,1]\} \\ \cup\{&(\infty, h_{12}, h_{12}, h_{21}, 0) \times (\infty, g_{12}, g_{12}h_{21}, h_{21}, 0); |h_{12}| = \infty, h_{21} \in (0, \overline{\kappa}],\ \&\ g_{12} \in sgn(h_{12})R_{+,\infty}\} \\ \cup\{&(\infty, h_{12}, h_{12}, 0, h_{22}) \times (g_{11}, h_{12}, g_{11}h_{22}, 0, h_{22}); |h_{12}| = \infty, h_{22} \in sgn(h_{12})(0,1],\ \&\ g_{11} \in R_{+,\infty}\} \end{aligned} \quad (8.42)$$

is always contained in GH_{II}.

For $j = III$, $h_{11} < \infty$, $|h_{12}| = \infty$, and

$$GH_{III} = \{(h_{11}, h_{12}, h_{11}h_{22}, 0, h_{22}) \times (0, h_{12}, 0, 0, h_{22}); h_{11} \geq 0 \ \& \ h_{22} \in sgn(h_{12})(0, 1]\}$$
$$\cup\{(h_{11}, h_{12}, 0, 0, 0) \times (0, g_{12}, 0, 0, 0); h_{11} \geq 0 \ \& \ g_{12} \in sgn(h_{12})R_{+,\infty}\}. \quad (8.43)$$

For $j = IV$, $h_{11} < \infty$, $|h_{12}| < \infty$, and

$$GH_{IV} = (R_+ \times R \times \{0\}^3) \times \{0\}^5. \quad (8.44)$$

Table IId contains information on the asymptotic size of the above tests when $k_2 = 5$ and $\alpha = \beta = .05$. Only results on subsampling upper and symmetric tests are reported. Results for lower and equal-tailed tests are virtually identical to the upper and symmetric ones, respectively, and FCV results have been reported earlier already. The asymptotic size of all tests equals 1. Note that the results do not depend on k_1.

In the rows, "Case I",...,"Case IV", the maximal null rejection probabilities are reported over the sets GH_I, $\widetilde{GH}_{II}$, GH_{III}, and GH_{IV}, respectively. For each column, the quantity $AsySz(\theta_0)$ is bounded from below by the maximum of the entries in this column in the rows above. Table IId shows that asymptotic null rejection probabilities equal to 1 occur in a wide array of parameter combinations that include the cases of weak and strong instruments and the case of weak and strong endogeneity of the regressor y_2. A priori, it is not clear that subsampling versions of the two-stage test have asymptotic size equal to 1. Note that, for example, in the linear IV model a two-sided symmetric confidence interval based on inverting a t-statistic has asymptotic confidence size equal to zero when based on normal critical values but has virtually correct asymptotic confidence size for subsampling critical values, see Dufour (1997) and AG (2005c) for details.

Table IId[20]

Maximal Null Rejection Probabilities of Subsampling Two-stage Test for Cases I-IV and $AsySz(\theta_0)$ for $k_2 = 5$ and $\alpha = \beta = .05$

Type\Case	I	II	III	IV	$AsySz(\theta_0)$
Upper	97.4	100	100	99.9	100
Symmetric	99.7	100	100	100	100

The results for subsampling tests immediately yield analogous results for m out of n bootstrap tests under the condition that $m^2/n \to 0$, where n and m are the sample size and the bootstrap blocksize, see AG (2005a) for a more detailed discussion of this point.

[20] The results in this table are based on $R = 50,000$ simulation repetitions.

9 Additional Applications

The asymptotic size properties of a two-stage test are investigated when the second stage test-statistic is robust to weak instruments in the case when the Hausman pretest rejects the pretest null hypothesis of regressor exogeneity. The asymptotic size of this modified two-stage test is shown to equal 1. In Subsection 9.2, a Hausman pretest of instrument exogeneity is considered as in Hahn, Ham, and Moon (2007). Severe size-problems occur.

9.1 Robust Second Stage Test Statistic

In order to show that the asymptotic size distortion of the two-stage test in (2.8) is not solely caused by the potential weakness of the instruments, this subsection studies the asymptotic size properties of a test based on a modification of the test statistic in (2.8) for the application in Section 2 that employs a second stage test statistic that is robust to weak instruments if the exogeneity hypothesis is rejected in the first stage. Specifically, the member $\mathcal{T}_g$ of the class of similar test statistics, introduced in Moreira's (2001) Example 2, that is robust to weak instruments, is used. Let

$$\begin{aligned} T_n^*(\theta_0) &= T_{OLS}^*(\theta_0) I(H_n \leq \chi_1^2(1-\beta)) + \mathcal{T}^*(\theta_0) I(H_n > \chi_1^2(1-\beta)), \\ \mathcal{T}^*(\theta) &= (n^{-1}\pi' Z^{\perp\prime} Z^{\perp}\pi)^{-1/2}\pi' n^{-1/2} Z^{\perp\prime}(y_1^{\perp} - y_2^{\perp}\theta). \end{aligned} \tag{9.45}$$

Define the two-stage test statistic $T_n(\theta_0)$ as $\pm T_n^*(\theta_0)$ or $|T_n^*(\theta_0)|$ depending on whether the test is a lower/upper one-sided or a symmetric two-sided test, respectively.[21] Define a modified estimator for the variance of u as

$$\widehat{\sigma}_u = \widehat{\sigma}_u(\widehat{\theta}_{OLS}) I(H_n \leq \chi_1^2(1-\beta)) + \widehat{\sigma}_u(\theta_0) I(H_n > \chi_1^2(1-\beta)). \tag{9.46}$$

As above in (6.14), consider rescaled versions of the test statistic. Instead of $\mathcal{T}^*(\theta_0)$ and $\widehat{\sigma}_u(\theta)$, $\mathcal{T}^{**}(\theta_0) = \mathcal{T}^*(\theta_0)/\sigma_u$ and $\widehat{\sigma}_u(\theta)/\sigma_u$ are used.

Jointly with the results in (6.23), (6.25), (6.29), and (6.31) we have in Cases I-IV under the null, $\mathcal{T}^{**}(\theta_0) \to_d s_{k_2}' \psi_{u,h_{22}}$ and $\widehat{\sigma}_u^2(\theta_0)/\sigma_u^2 \to_p 1$. In all Cases I-IV, let η_h be

[21] To avoid additional subindices, the same notation, $T_n^*(\theta_0)$, is used for both statistics in (9.45) and (2.8). Similarly, in (9.46), the same notation, $\widehat{\sigma}_u$, is used for the modified variance estimator as was used in (2.9). The same is true for η_h and other quantities.

The second stage test statistic $\mathcal{T}^*(\theta)$ is an infeasible version of the similar statistic $\mathcal{T}_g$ in Example 2 in Moreira (2001) with $g \equiv \pi$, because π is unknown. The goal here is to establish that a robust second stage statistic - in the case where the Hausman pretest rejects - does not solve the size distortion problem of the two stage test. Using $g \equiv \widehat{\pi}$, where $\widehat{\pi}$ is the restricted maximum likelihood estimator of π when the structural parameter vector is fixed at the null values, or $g = e$, where e is a k_2 vector of ones, provide feasible alternatives. However, it is more difficult to handle the asymptotic results in these latter cases because the result $\mathcal{T}^{**}(\theta_0) \to_d s_{k_2}' \psi_{u,h_{22}}$ would no longer hold jointly with the results in (6.23), (6.25), (6.29), and (6.31): a different "direction" s_{k_2} would arise in this case that depends on π. This would unnecessarily complicate the evaluation by simulation of the formulas in the asymptotic size results of Theorem 3.

defined as above with its first component replaced by $s_{k_2}'\psi_{u,h_{22}}$ and let ξ_h be defined as above with the fifth component replaced by 1. With these modifications, the result $T_n^{**}(\theta_0) \rightarrow_d J_h^{**}$ in (6.32) and the results in Theorem 3 still hold.

Table IIe contains information on the maximal null rejection probabilities in Cases I-IV of the tests in this subsection when $k_2 = 5$ and $\alpha = \beta = .05$. Only results for upper and symmetric FCV and subsampling tests are reported because lower and equal-tailed tests and hybrid tests have essentially the same size properties. Table IIe shows that $AsySz(\theta_0) = 1$ for all types of FCV and subsampling tests. The maximal null rejection probabilities essentially equal 1 on each of the subsets defined by Cases I-IV. Therefore, use of a robust second stage statistic in the case where the Hausman pretest rejects the null hypothesis, does not alleviate the problem of size distortion.

Table IIe[22]

Maximal Null Rejection Probabilities of Two-stage Test in (9.45) for Cases I-IV and $AsySz(\theta_0)$ for $k_2 = 5$ and $\alpha = \beta = .05$

	Upper		Symmetric	
	Sub	FCV	Sub	FCV
Case I	97.4	97.4	99.9	100
Case II	100	97.5	100	100
Case III	100	99.8	100	100
Case IV	99.8	99.8	99.9	100
$AsySz(\theta_0)$	100	99.8	100	100

9.2 Pretesting Instrument Exogeneity

In this application, the Hausman pretest is used to test for instrument exogeneity. More precisely, the instruments are decomposed into $Z = (W, S)$, where W has k_{21} and S has k_{22} columns and $k_2 = k_{21} + k_{22}$. The instruments S are potentially invalid, that is correlated with u, while the instruments W are assumed to be valid. The Hausman pretest tests whether S are valid instruments. If the pretest is rejected, then in the second stage, the hypothesis $H_0 : \theta = \theta_0$ is tested by using a t-statistic (or alternatively, the similar test statistics $\mathcal{T}_g$; which one of the two is used does not matter for the results below) based on only the instruments W. Otherwise, a t-statistic (or similar statistic) is used based on all instruments Z. In an application, one could think of W and S as weak and strong instruments, respectively.

To test orthogonality of the instruments S, two different versions of a pretest are being considered. The first one, denoted again by H_n, is the standard Hausman test and the second one, denoted by $\mathcal{H}_4$ is the version of the Hausman test introduced in Hahn, Ham, and Moon (2007) using their notation. In this subsection, for ease of presentation, there are no included exogenous variables and θ is again scalar. The

[22]The results in this table are based on $R = 50,000$ simulation repetitions.

formulas are

$$\begin{aligned} H_n &= \frac{n(\widehat{\theta}_W - \widehat{\theta}_Z)^2}{\widehat{V}_W - \widehat{V}_Z}, \\ \mathcal{H}_4 &= \widetilde{\sigma}_u^{-2}(y_1 - y_2\widehat{\theta}_Z)'W[W'W - W'y_2(y_2'P_Z y_2)^{-1}y_2'W]^{-1}W'(y_1 - y_2\widehat{\theta}_Z), \\ \widetilde{\sigma}_u^2 &= n^{-1}(y_1 - y_2\widehat{\theta}_Z)'M_Z(y_1 - y_2\widehat{\theta}_Z), \end{aligned} \tag{9.47}$$

where $\widehat{\theta}_W$ and $\widehat{V}_W$ are defined analogously to the 2SLS expressions in (2.6), when the estimators are based only on the instruments W; likewise, $\widehat{\theta}_Z$ and $\widehat{V}_Z$ denote what was previously denoted by $\widehat{\theta}_{2SLS}$ and $\widehat{V}_{2SLS}$ in (2.6). Similar straightforward modifications to the notation are used for other expressions, e.g. $T_W^{**}(\theta_0)$ and $T_Z^{**}(\theta_0)$ are used in place of $T_{2SLS}^{**}(\theta_0)$ when the statistic is based on instruments W or Z, respectively. As shown by Hahn, Ham, and Moon (2007), $\mathcal{H}_4$ is asymptotically χ^2 even when instruments are weak. This is not true for H_n.

For simplicity, assume $k_{22} = 1$, $ES_i = 0$, and $ES_iW_i = 0_{k_{21}}$, where 0_k denotes a k-dimensional vector of zeros. That is, there is only one (potentially) strong instrument, it has mean zero and is uncorrelated with the other instruments. Simply view S_i as the residual of a strong instrument that is being regressed on the instruments W_i.

Denote by $\{\gamma_{n,h}\} \subset \Gamma$ a sequence of parameters with components $\gamma_{n,h,1}, \gamma_{n,h,2}$, and $\gamma_{n,h,3}$, such that

$$\begin{aligned} \gamma_{n,h,1} &= (||((E_{F_n}Z_iZ_i')^{1/2}\pi_n/(E_{F_n}v_i^2)^{-1/2}||, Corr_{F_n}(u_i, S_i)), \\ \gamma_{n,h,2} &= (\gamma_{n,h,1}, Corr_{F_n}(u_i, v_i)),\ n^{1/2}\gamma_{n,h,1} \to h_1,\ \gamma_{n,h,2} \to h_2, \text{ and} \end{aligned} \tag{9.48}$$

$\gamma_{n,h,3}$ satisfying similar restrictions to those in (6.8) including $E_{F_n}W_iS_iu_i^2 = E_{F_n}W_iS_iu_i^2 = 0_{k_{21}}$, and $var_{F_n}S_iu_i/(E_{F_n}S_i^2E_{F_n}u_i^2) = 1 + Corr_{F_n}^2(S_i, v_i)$. With these assumptions, under any sequence $\{\gamma_{n,h}\}$, the following convergence result similar to (6.20) holds

$$\begin{aligned} &\left(\begin{array}{c} (n^{-1}Z'Z)^{-1/2}n^{-1/2}((W'u)', S'u - ES'u)'/\sigma_u \\ (n^{-1}Z'Z)^{-1/2}n^{-1/2}Z'v/\sigma_v \end{array} \right) \to_d \left(\begin{array}{c} \psi_{u,h_2} \\ \psi_{v,h_2} \end{array} \right) \\ &\sim N(0, \left[\begin{array}{cc} I_{k_2,h_{22}} & h_{23}I_{k_2} \\ h_{23}I_{k_2} & I_{k_2} \end{array} \right]) \text{ for } I_{k_2,h_{22}} = \left[\begin{array}{cc} I_{k_{21}} & 0 \\ 0 & 1 + h_{22}^2 \end{array} \right]. \end{aligned} \tag{9.49}$$

For simplicity, it is only shown that conditional on the Hausman pretest (based on H_n or $\mathcal{H}_4$) not rejecting the pretest null hypothesis of instrument exogeneity, the asymptotic size of the two-stage FCV procedure equals 1. It turns out that, to show this, only requires looking at a particular scenario modelled by

$$\begin{aligned} &y_2 = n^{-1/4}\pi_S S + v \\ &n^{1/2}\gamma_{n,h,1,2} \to h_{12} \text{ finite} \end{aligned} \tag{9.50}$$

for a fixed nonzero number π_S.[23] The coefficients on the weak instruments are modelled as zero while the coefficient on the strong instrument shrinks to zero at rate $n^{-1/4}$. The instruments are strong in the sense that the concentration parameter goes off to infinity. Because $n^{1/2}\gamma_{n,h,1,2} \to h_{12}$ for h_{12} finite, the instrument is weakly endogenous. Model (9.50), when viewed as a sequence $\{\gamma_{n,h}\}$, has $h = (h_1, h_2)$ with $h_1 = (\infty, h_{12})$ and $h_2 = (0, 0, h_{23})$.

Using (9.49) it follows that under (9.50) for $\xi_h = (\xi_{1,h}, ..., \xi_{6,h})'$

$$\begin{pmatrix} n^{-1/4}y_2'P_Z u/(\sigma_u\sigma_S) \\ y_2'P_W u/(\sigma_u\sigma_v) \\ n^{-1/2}y_2'P_Z y_2/\sigma_S^2 \\ y_2'P_W y_2/\sigma_v^2 \\ \widehat{\sigma}_u^2(\widehat{\theta}_Z)/\sigma_u^2 \\ \widehat{\sigma}_u^2(\widehat{\theta}_W)/\sigma_u^2 \end{pmatrix} \to_d \xi_h \sim \begin{pmatrix} \pi_S(\psi_{u,h_2,k_2} + h_{12}) \\ \psi'_{v,h_2,1:k_{21}}\psi_{u,h_2,1:k_{21}} \\ \pi_S^2 \\ \psi'_{v,h_2,1:k_{21}}\psi_{v,h_2,1:k_{21}} \\ 1 \\ 1 - 2h_{23}\xi_{2,h}/\xi_{4,h} + (\xi_{2,h}/\xi_{4,h})^2 \end{pmatrix}, \quad (9.51)$$

where ψ_{u,h_2,k_2} denotes the k_2-th entry in ψ_{u,h_2} and $\psi_{v,h_2,1:k_{21}}$ and $\psi_{u,h_2,1:k_{21}}$ denote the first k_{21} entries of ψ_{v,h_2} and ψ_{u,h_2}, respectively. The statistics $\widehat{\theta}_W$ and $\widehat{V}_W$ in H_n are of higher order than the statistics $\widehat{\theta}_Z$ and $\widehat{V}_Z$ and the latter hence do not matter for the asymptotic distribution of H_n in model (9.50). Likewise, in $\mathcal{H}_4$, $W'W$ is of higher order than $W'y_2(y_2'P_Z y_2)^{-1}y_2'W$ and the latter term can be neglected for the limit theory. Finally, $\widetilde{\sigma}_u^2/\sigma_u^2 \to_p 1$. By (9.51), we therefore have in model (9.50)

$$\begin{pmatrix} T_Z^{**}(\theta_0) \\ H_n \\ \mathcal{H}_4 \end{pmatrix} \to_d \eta_h = \begin{pmatrix} \psi_{u,h_2,k_2} + h_{12} \\ \xi_{2,h}^2\xi_{4,h}/(\xi_{4,h}^2 - 2h_{23}\xi_{2,h}\xi_{4,h} + \xi_{2,h}^2) \\ \psi'_{u,h_2,1:k_{21}}\psi_{u,h_2,1:k_{21}} \end{pmatrix}. \quad (9.52)$$

Let $\eta_h = (\eta_{1,h}, \eta_{2,h}, \eta_{3,h})'$. Note that $\eta_{1,h}$ and $\eta_{2,h}$ are independent, because $\xi_{2,h}$ and $\xi_{4,h}$ only depend on $\psi_{v,h_2,1:k_{21}}$ and $\psi_{u,h_2,1:k_{21}}$ and by (9.49) those random variables are independent of ψ_{u,h_2,k_2}. Therefore, asymptotically, conditional on the pretest based on H_n not rejecting, $T_Z^{**}(\theta_0)$ is distributed as $\psi_{u,h_2,k_2} + h_{12}$. Hence, picking h_{12} large enough (or small enough for lower one-sided tests), it is clear that the conditional size of the two-stage test, conditional on the pretest based on H_n not rejecting the pretest null hypothesis, is 1 asymptotically. The same argument holds for the two-stage test with the pretest based on $\mathcal{H}_4$. The limit distribution of $\mathcal{H}_4$ is a chi-squared with k_{21} degrees of freedom that is independent of $\psi_{u,h_2,k_2} + h_{12}$.

[23] The result of conditional size equal to 1 asymptotically can be shown by looking at many different sequences of the nuisance parameters. Here, I pick one particularly simple choice that makes the analysis easy. Assume, in addition, that $\sigma_v^2 = E_{F_n}v_i^2$ and $\sigma_S^2 = E_{F_n}S_i^2$ are nonzero and do not depend on n.

Additional References

Dufour, J.M. (1997): "Some Impossibility Theorems in Econometrics with Applications to Structural and Dynamic Models", *Econometrica* 65, 1365–1388.

Guggenberger, P. and M. Wolf (2004): "Subsampling tests of parameter hypotheses and overidentifying restrictions with possible failure of identification," *unpublished working paper, Dept. Econom., UCLA.*

TABLE Ic[24]

Finite Sample Power of Symmetric Two-stage Test and 2SLS Based t-Test

$\theta = .1; \theta_0 = 0; k_1 = 0$, $\alpha = \beta = .05$, $n = 1000$, $k_2 = 5$; based on 50,000 repetitions

$\mu^2 \backslash \rho$	0	.05	.1	.2	.3	.4	.5	.6
0	88.2;0.0	99.6;0.1	99.9;0.4	99.9;1.8	99.9;6.2	99.9;16.0	99.9;32.2	99.9;52.0
13	88.0;2.3	98.2;3.6	97.7;5.0	95.2;8.8	91.0;14.1	83.8;19.8	71.8;26.0	56.3;32.4
50	88.3;9.9	95.7;11.1	91.8;12.4	76.0;15.2	49.4;18.0	25.6;20.8	21.2;24.0	23.5;27.1
113	89.6;18.5	93.3;19.6	84.5;20.5	50.4;22.7	25.4;24.9	25.5;27.1	27.3;29.1	29.0;31.2
200	91.3;29.6	91.0;30.4	76.3;31.2	36.9;32.7	33.6;34.3	35.0;35.9	36.4;37.5	37.8;39.0
313	93.0;42.7	88.7;43.2	68.8;43.7	44.8;44.8	45.5;45.8	46.5;46.8	47.5;47.8	48.4;48.9
450	94.5;56.0	87.0;56.3	67.5;56.6	57.4;57.2	58.0;57.8	58.5;58.3	59.1;58.9	59.7;59.4
613	95.8;69.3	87.0;69.3	73.1;69.4	69.7;69.5	70.0;69.6	70.2;69.8	70.5;70.0	70.8;70.1

TABLE Ic (continued)

Finite Sample Power of Symmetric Two-stage Test and 2SLS Based t-Test

$\theta = .2; \theta_0 = 0; k_1 = 0$, $\alpha = \beta = .05$, $n = 1000$, $k_2 = 5$; based on 50,000 repetitions

$\mu^2 \backslash \rho$	0	.05	.1	.2	.3	.4	.5	.6
0	99.8;0.3	99.9;0.7	99.9;1.5	99.9;4.7	99.9;12.0	99.9;24.7	99.9;41.8	99.9;60.0
13	98.8;8.5	98.4;10.8	97.6;13.5	95.2;19.4	91.0;25.8	83.8;32.0	72.9;38.3	61.4;44.4
50	97.9;30.9	95.7;32.5	91.6;34.0	76.0;37.1	53.2;40.0	43.3;42.9	44.1;45.7	46.5;48.3
113	97.8;57.3	93.4;57.8	84.6;58.3	63.2;59.3	60.7;60.4	61.7;61.4	62.7;62.4	63.8;63.4
200	97.3;80.3	92.0;80.1	84.3;80.0	80.5;79.9	80.6;79.8	80.7;79.7	80.8;79.6	81.0;79.7
313	98.2;93.7	95.2;93.5	93.7;93.3	93.4;92.8	93.1;92.4	92.6;92.0	92.6;91.6	92.4;91.3
450	99.3;98.6	98.6;98.4	98.5;98.4	98.3;98.1	98.2;97.9	98.1;97.7	97.9;97.5	97.8;97.2
613	99.9;99.8	99.8;99.8	99.8;99.7	99.7;99.7	99.7;99.6	99.6;99.5	99.6;99.4	99.5;99.3

Table Id[25]

Finite Sample Rejection Probabilities for the Test in Section 2

$k_1 = 0$, $k_2 = 1$, $\alpha = \beta = .05$

I. Case I setup: $\rho = 10n^{-1/2}; \pi = n^{-1/4}$

n	π	ρ	Upper	Sym	HPre	CondlUpper	CondlSym
100	.32	1	25.3	22.1	86.6	100	100
1000	.18	.32	60.7	61.3	39.3	100	100
10000	.10	.10	84.5	86.5	15.6	100	100
100000	.06	.03	92.3	94.7	8.1	100	100

II. Case II setup: $\rho = n^{-1/4}; \pi = n^{-1/4}$

[24] For each entry in the table, the first component is the finite sample null rejection probability of the two-stage test and the second component is the null rejection probability of the t-test based on 2SLS.

[25] The simulation results are based on $R = 10,000$ simulation repetitions.

n	π	ρ	Upper	Sym	HPre	CondlUpper	CondlSym
100	.32	.32	88.3	83.2	4.5	92.7	87.2
1000	.18	.18	86.5	87.6	13.5	100	100
10000	.10	.10	84.5	86.5	15.6	100	100
100000	.06	.06	83.9	86.3	16.2	100	100

III. Case III setup $\rho = n^{-1/4}; \pi = 10n^{-1/2}$

n	π	ρ	Upper	Sym	HPre	CondlUpper	CondlSym
100	1.00	.32	30.7	27.7	59.2	74.1	63.1
1000	.32	.18	60.8	62.7	39.2	100	100
10000	.10	.10	84.5	86.5	15.6	100	100
100000	.03	.06	92.6	94.9	7.7	100	100

IV. Case IV setup: $\rho = 10n^{-1/2}; \pi = 10n^{-1/2}$

n	π	ρ	Upper	Sym	HPre	CondlUpper	CondlSym
100	1	1	7.7	5.3	100	-	-
1000	.32	.32	12.5	13.5	88.2	100	100
10000	.10	.10	84.5	86.5	15.6	100	100
100000	.03	.03	95.4	97.7	5.5	100	100

I'. Case I setup: $\rho = 10n^{-1/2}; \pi = 1$

n	π	ρ	Upper	Sym	HPre	CondlUpper	CondlSym
100	1	1	7.7	5.3	100	-	-
1000	1	.32	4.9	4.8	100	-	-
10000	1	.10	5.0	4.9	100	-	-
100000	1	.03	5.1	4.8	100	-	-

II'. Case II setup: $\rho = .2; \pi = 1$

n	π	ρ	Upper	Sym	HPre	CondlUpper	CondlSym
100	1	.2	31.1	24.0	25.4	41.7	29.6
1000	1	.2	4.9	5.1	99.5	100	100
10000	1	.2	5.0	4.9	100	-	-
100000	1	.2	5.1	4.8	100	-	-