Ec 501
Game Theory

Trembling-Hand Perfect Nash Equilibrium

Let G be any finite normal form game. A strategy $\sigma_i \in \Sigma_i$ is *totally mixed strategy* if $\sigma_i(s_i) > 0$ for all $s_i \in S_i$.

A strategy profile σ is a *trembling-hand perfect Nash equilibrium* if there exist a sequence of totally mixed strategy profiles σ^n converging to σ such that $\sigma_i \in B_i(\sigma^n_{-i})$ for all n.

Fact: *Every trembling-hand perfect Nash equilibrium is a Nash equilibrium.*

Proof: Let σ be a trembling-hnad perfect Nash equilibrium. Pick any $\sigma'_i \in \Sigma_i$. Note that $U_i(\sigma_i, \sigma^n_{-i}) - U_i(\sigma'_i, \sigma^n_{-i}) \geq 0$ for all n. Then, continuity of U_i implies $U_i(\sigma_i, \sigma_{-i}) - U_i(\sigma'_i, \sigma_{-i}) \geq 0$ and therefore $\sigma_i \in B_i(\sigma_{-i})$ for all $i \in I$. $\square$

Theorem: *Every finite normal form game has a trembling-hand perfect Nash equilibrium.*

Proof: Pick a profile σ^0 of totally mixed strategies. Pick also a sequence of $\epsilon^n_i \in (0,1)$ converging to 0, for each i. Define, $u^n_i(s) := U_i((\epsilon^n_i \sigma^0_i + (1-\epsilon^n_i)s_i)_{i \in I})$ for all $s \in S$. Let G^n be the game obtained from G when each u_i is replaced with the corresponding u^n_i. By the Nash equilibrium existence theorem, there exists a Nash equilibrium σ^n for each G^n. Since each Σ_i is compact, there exists a convergent subsequence of σ^n. Without loss of generality, assume this subsequence is the sequence itself and let σ be its limit. We will conclude the proof by showing that for some N, $\sigma_i \in B_i(\sigma^n_{-i})$ for all $i \in I$ and $n \geq N$. Then, eliminating the first n elements of the sequence σ^n and renumbering yields a sequence such that $\sigma_i \in B_i(\sigma^n_{-i})$ for all $i \in I$.

For each s_i such that $\sigma_i(s_i) > 0$, there exists an integer $N(s_i)$ such that $\sigma^n_i(s_i) > 0$ for all $n \geq N(s_i)$. Let N be the maximum of these $N(s_i)$ over all $s_i \in S_i$ and $i \in I$. Then, for all s_i such that $\sigma_i(s_i) > 0$, $s_i \in B_i(\sigma^n_{-i})$ for all $n \geq N$. Hence, $\sigma_i \in B_i(\sigma^n_{-i})$ for all $n \geq N$ as desired. $\square$

Exercise: Make the following claim precise and either prove it or provide a counter example. If we had used the construction used in the proof of the previous proof as the definition of trembling-hand perfect Nash equilibrium nothing would have changed.