Trembling-Hand Perfect Nash Equilibrium

Let \( G \) be any finite normal form game. A strategy \( \sigma_i \in \Sigma_i \) is totally mixed strategy if \( \sigma_i(s_i) > 0 \) for all \( s_i \in S_i \).

A strategy profile \( \sigma \) is a trembling-hand perfect Nash equilibrium if there exists a sequence of totally mixed strategy profiles \( \sigma^n \) converging to \( \sigma \) such that \( \sigma_i \in B_i(\sigma^n_i) \) for all \( n \).

Fact: Every trembling-hand perfect Nash equilibrium is a Nash equilibrium.

Proof: Let \( \sigma \) be a trembling-hand perfect Nash equilibrium. Pick any \( \sigma'_i \in \Sigma_i \). Note that \( U_i(\sigma_i, \sigma^n_i) - U_i(\sigma'_i, \sigma^n_i) \geq 0 \) for all \( n \). Then, continuity of \( U_i \) implies \( U_i(\sigma_i, \sigma_{-i}) - U_i(\sigma'_i, \sigma_{-i}) \geq 0 \) and therefore \( \sigma_i \in B_i(\sigma_{-i}) \) for all \( i \in I \).

Theorem: Every finite normal form game has a trembling-hand perfect Nash equilibrium.

Proof: Pick a profile \( \sigma^0 \) of totally mixed strategies. Pick also a sequence of \( \epsilon^n_i \in (0, 1) \) converging to 0, for each \( i \). Define, \( u^n_i(s) := U_i((\epsilon^n_i\sigma^0_i + (1 - \epsilon^n_i)s_i)_{i \in I}) \) for all \( s \in S \). Let \( G^n \) be the game obtained from \( G \) when each \( u_i \) is replaced with the corresponding \( u^n_i \). By the Nash equilibrium existence theorem, there exists a Nash equilibrium \( \sigma^n \) for each \( G^n \). Since each \( \Sigma_i \) is compact, there exists a convergent subsequence of \( \sigma^n \). Without loss of generality, assume this subsequence is the sequence itself and let \( \sigma \) be its limit. We will conclude the proof by showing that for some \( N \), \( \sigma_i \in B_i(\sigma^n_{-i}) \) for all \( i \in I \) and \( n \geq N \). Then, eliminating the first \( n \) elements of the sequence \( \sigma^n \) and renumbering yields a sequence such that \( \sigma_i \in B_i(\sigma^n_{-i}) \) for all \( i \in I \).

For each \( s_i \) such that \( \sigma_i(s_i) > 0 \), there exists an integer \( N(s_i) \) such that \( \sigma^n_i(s_i) > 0 \) for all \( n \geq N(s_i) \). Let \( N \) be the maximum of these \( N(s_i) \) over all \( s_i \in S_i \) and \( i \in I \). Then, for all \( s_i \) such that \( \sigma_i(s_i) > 0 \), \( s_i \in B_i(\sigma^n_{-i}) \) for all \( n \geq N \). Hence, \( \sigma_i \in B_i(\sigma^n_{-i}) \) for all \( n \geq N \) as desired.

Exercise: Make the following claim precise and either prove it or provide a counter example.

If we had used the construction used in the proof of the previous proof as the definition of trembling-hand perfect Nash equilibrium nothing would have changed.