Normal Form Games

1 Definitions

s_i is a pure strategy or action for player $i \in I := \{1, 2, ..., L\}$

$S_i = \{s_i^1, s_i^2, ..., s_i^{k_i}\}$ is the set of all pure strategies for player i

$\sigma_i : S_i \to [0, 1]$, where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$ is a mixed strategy for player $i \in I$

$\sum_i$ is the set of all mixed strategies for player i.

Sometimes we view a mixed strategy as an element of $\mathbb{R}_+^{k_i}$, where $x \in \mathbb{R}_+^{k_i}$ denotes the $\sigma_i \in \sum_i$ such that $\sigma_i(s_i^j) = x_j$ for all $j = 1, 2, ..., k_i$

$s := (s_1, ..., s_L)$ is a pure strategy profile

$S := \times_{i \in I} S_i$ is the set of all pure strategy profiles

$\sigma := (\sigma_1, ..., \sigma_N)$ is a mixed strategy profile

$\sum := \times_{i \in I} \sum_i$ is the set of all mixed strategy profiles

$S_{-i} := \times_{j \in I \backslash \{i\}} S_j$ is the set of pure strategy profiles for players in $I \backslash \{i\}$

$\sum_{-i} := \times_{j \in I \backslash \{i\}} \sum_j$ is the set of all mixed strategy profiles for players in $I \backslash \{i\}$

We identify $s = (s_1, ..., s_n)$ with (s_i, s_{-i}) and $(\sigma_1, ..., \sigma_n)$ with (σ_i, σ_{-i}) in spite of the fact that the $s_j's$ are not in the right sequence.

When objects like s, s_i, s_j, s_{-i} (or $(\sigma, \sigma_i, \sigma_j, \sigma_{-i})$ appear in the same sentence, it is understood that the s_{-i} refers to the corresponding $n-1$ entries of s and that s_i is the ith entry of s and s_j is the jth entry of s and/or s_{-i}. When s_i may be different than the corresponding entry of s, we use $\widehat{s}_i$ or $\overline{s}_i$ or $\widetilde{s}_i$ instead of s_i. Similar statements hold for mixed strategies and mixed strategy profiles.

A utility function or a payoff for player i is a function $u_i : S \to \mathbb{R}$. The function u_i is interpreted as a von Neumann-Morgenstern utility index for player i on the set of prizes S. Sometimes it is better to think of a set of prizes X and a function $f : S \to X$ as determining the outcome of the game. In this view, players have preferences over lotteries over prizes, pure strategy profiles yield a prize, mixed strategy profiles yield a lottery over prizes (how?) and players evaluate these lotteries according to the expected utility function.

Hence,

$$U_i(\sigma) := \sum_{s \in S} u_i(s) \cdot \times_{i \in I} \sigma_i(s_i)$$

or in the second formulation

$$U_i(\sigma) := \sum_{s \in S} u_i\left(f(x)\right) \cdot \times_{i \in I} \sigma_i(s_i)$$

Exercise 1: Show that $U_i(\alpha\sigma_i + (1-\alpha)\sigma_i', \sigma_{-i}) = \alpha U_i(\sigma_i, \sigma_{-i}) + (1-\alpha)U_i(\sigma_i', \sigma_{-i})$

A normal form game or a finite normal form game is the collection $G := \{(S_i, u_i)_{i \in I}\}$.

2 Best Responses

A strategy $\sigma_i \in \sum_i$ is a *best response* to σ_{-i} if

$$U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \sum_i$.

Let $B_i(\sigma_{-i})$ denote the set of all best responses to $\sigma_{-i} \in \sum_{-i}$.

Exercise 2: Show that $B_i(\sigma_{-i}) \neq \emptyset$, for all $\sigma_{-i} \in \sum_{-i}$

Exercise 3: Show that $B_i(\sigma_{-i})$ is a closed set.

A strategy profile is a Nash equilibrium of the game G if $\sigma_i \in B_i(\sigma_{-i})$ for all $i \in I$.

Theorem: *Every normal form game G has at least one Nash equilibrium.*

We will not prove the theorem above, but we will discuss Kakutani's Fixed Point Theorem which is the key step in the proof.

Let $X \subset \mathbb{R}^k, Y \subset \mathbb{R}^l$ be arbitrary sets. A *correspondence* F from X to Y is a function that associates a non-empty subset of Y with each element of X.

The correspondence F is upper hemi-continuous if for every sequence $\{x_n\}$ in X, $[y_n \in F(x_n)$ for all n, $\{x_n\}$ converges to x, y_n converges to $y]$ implies $y \in F(x)$.

Fact: *If Y is compact and the upper hemi-continuous correspondence F from X to Y is also a function, that is, if $F(x)$ is a singleton for every $x \in X$, then F viewed as a function is continuous.*

Fact: *A correspondence F from X to Y is upper hemi-continuous iff and only if the graph of F, defined to be the set $\{(x, y)|x \in X, y \in F(x)\}$, is closed.*

The correspondence F from X to Y is convex-valued if the set $F(x)$ is convex for every $x \in X$. If $Y = X$, then an $x \in X$ such that $x \in F(x)$ is called a fixed-point of F.

The following result is known as Kakutani's Fixed Point Theorem:

Theorem: Let $X \subset \mathbb{R}^n$ be a compact, convex set and let F be a convex-valued, upper hemi-continuous correspondence from X to X. Then there exists $x \in X$ s.t. $x \in F(x)$.

Exercise 5: Let $X \subset \mathbb{R}^n$ be a compact convex set. Let g be a continuous function from X to X such that $g(x) = x$ for all $x \in g(X)$ and let F be a convex-valued upper hemi-continuous correspondence from $g(X)$ to $g(X)$. Prove that F has a fixed point.

To see how Kakutani's Fixed-Point Theorem ensures the existence of a Nash equilibrium, define the correspondence F_i from $\sum$ to $\sum_i$ as follows:

$$F_i(\sigma) = B_i(\sigma_{-i}) \text{ for all } \sigma \in \sum$$

Then, define the correspondence F from $\sum$ to $\sum$ as follows:

$$F(\sigma) = \times_{i \in I} F_i(\sigma) \text{ for all } \sigma \in \sum$$

Note that $\sum$ is a compact convex subset of $\mathbb{R}$. Verify that each F_i is convex-valued and upper hemi-continuous (see the next handout) and hence F is convex-valued and upper hemi-continuous. Then, F has a fixed-point. Observe that this fixed point is a Nash equilibrium.